

Hard Thermal Loops, Weak Gravitational Fields and The Quark Gluon Plasma Energy Momentum Tensor.

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Abstract

We use an auxiliary field construction to discuss the hard thermal loop effective action associated with massless thermal $SU(N)$ QCD interacting with a weak gravitational field. It is demonstrated that the previous attempt to derive this effective action has only been partially successful and that it is presently only known to first order in the graviton coupling constant. This is still sufficient to enable a calculation of a symmetric traceless quark gluon plasma energy momentum tensor. Finally, we comment on the conserved currents and charges of the derived energy momentum tensor.

1 Introduction

There has been much discussion on various ways of deriving the energy momentum tensor of the high temperature deconfined massless quark gluon plasma (QGP), which is non-standard due to the non-local nature of the perturbative resummation required to describe the plasma even at the lowest orders of g , the QCD coupling constant [1]. Weldon [2] describes the non-local dynamics of the QGP using a lagrangian with a local dependence on auxiliary fields (rather than a non local dependence on physical fields). He then proceeds to derive the QGP energy momentum tensor by calculating the canonical energy momentum tensor of this lagrangian (which however is neither symmetric nor traceless despite the fact that the (classical) field theory of the massless QGP is scale invariant). Blaizot and Iancu [3] deduce three forms of the QGP energy momentum tensor for a pure gluon plasma by integrating the divergence condition $\partial_\nu T^{\mu\nu} = j^\mu$ (where j^μ is a current due to an external source). This approach however does not display the link between symmetries of the theory and the conserved quantities constructed from the energy momentum tensor.

Brandt, Frenkel and Taylor [6] discuss the interaction of the QGP with a weak gravitational field and introduce the two effective actions $\Gamma^{HTL}[A, g]$ and $\Gamma^{HTL(quark)}[A, e]$, the former for gluon graviton interactions and the latter for the interaction of quarks with gluons and vierbiens fields (where A, g, e are (classical) gluon, metric, and vierbien fields). They define the QGP energy momentum tensor to be the sum of the two expressions

$$T_{\alpha\beta}^{HTL} \equiv 2\lim_{g \rightarrow \eta} \left(\frac{\delta \Gamma^{HTL}[A, g]}{\delta g^{\alpha\beta}} \right) \quad T_{\alpha\beta}^{HTL(quark)} \equiv \lim_{e \rightarrow \eta} \left(e^a_\alpha \frac{\delta \Gamma^{HTL(quark)}[A, e]}{\delta e^{a\beta}} \right)$$

which are the definitions of $T_{\alpha\beta}^{HTL}$ and $T_{\alpha\beta}^{HTL(quark)}$ adopted in this paper. Note that the above definition of $T_{\alpha\beta}^{HTL}$ is equivalent to twice the leading temperature contribution of the sum over n of thermal Feynman graphs consisting of a graviton current insertion $T_{\alpha\beta}^{\{T=0\}}$ and n gluon current insertions. ($T_{\alpha\beta}^{\{T=0\}}$ is the usual gluon contribution to the zero temperature QCD energy momentum tensor. It is also the zero temperature coupling to the graviton field, $\varphi^{\alpha\beta}$, which is related to the metric by $g^{\alpha\beta} = \eta^{\alpha\beta} + \kappa \varphi^{\alpha\beta}$ where κ is the graviton coupling constant). A similar interpretation can be given for the quark tensor.

The properties of the energy momentum tensor such as divergencelessness (unlike Blaizot and Iancu we do not have external sources), tracelessness and symmetry (on use of the equations of motion if need be) are directly linked to the symmetries of the hard thermal loop theory. These properties can be deduced either from the invariances of the above effective actions or, equivalently, from the properties of hard thermal loop diagrams with graviton insertions.

In this paper the two effective actions above are written, to first order in the graviton coupling constant $\kappa \ll 1$, in terms of local actions with auxiliary fields. This greatly simplifies the calculation of $T_{\alpha\beta}^{HTL}$ and $T_{\alpha\beta}^{HTL(quark)}$. We note that a traceless and symmetric energy momentum tensor including a contribution due to the presence of quarks has not previously been written to all orders in g in the literature. Furthermore, the auxiliary field construction explicitly shows that the gluon hard thermal loop effective action (with no gravitons), together with the symmetry properties of the gluon graviton effective actions

given in [6] (and by conditions (A) \rightarrow (D) of section (2.2) below), are *not* sufficient to determine the gluon graviton effective action $\Gamma^{HTL}[A, g]$, even at $O(\kappa)$. This is contrary to assertion made in [6]. We give supplementary conditions that enable determination of $\Gamma^{HTL}[A, g]$ to $O(\kappa)$. Further, it is shown that both $\Gamma^{HTL}[A, g]$ and $\Gamma^{HTL(quark)}[A, g]$ are not known at present to $O(\kappa^2)$.

We show that the derived gluon contribution to the energy momentum tensor, $T_{\alpha\beta}^{HTL}$, gives an integrated energy density (i.e. $P_0 = \int d^3x T_{00}^{HTL}$) which is a positive definite functional of the gauge fields, although this property is not manifest. This is a straightforward corollary of the results of Blaizot and Iancu [3]. Finally, it is shown that the gluon contribution to the integrated energy density is also given by the Hamiltonian associated with the flatspace local auxiliary field action on elimination of the auxiliary fields.

2 Gluon fields

2.1 Flat space Gluon auxiliary fields

The notation used in this paper is as follows: we consider the interaction of a weak external graviton field with massless thermal SU(N) QCD in the deconfined phase at temperature T and zero chemical potential with N_f quark flavours. The generators of the fundamental and adjoint representations of the colour group are denoted respectively by t^A and T^A . These are taken to be hermitian and are normalised by

$$\text{tr}(t^A t^B) = \frac{1}{2} \delta^{AB} \quad \text{tr}(T^A T^B) = N \delta^{AB} \quad (2.1)$$

The structure constants f^{ABC} satisfy $[t^A, t^B] = +if^{ABC}t^C$ and $(T^A)_{BC} = -if^{ABC}$. We have quark fields ψ , gauge fields in the two above representations $A_\mu = A_\mu^A t^A$, $\mathcal{A}_\mu = A_\mu^A T^A$, vierbein fields $e_{a\mu}$ where a denotes a local Lorentz index and graviton fields $\phi^{\alpha\beta}$ given by

$$g^{\alpha\beta} = \eta^{\alpha\beta} + \kappa \phi^{\alpha\beta} \quad (2.2)$$

where $\kappa \ll 1$.

Initially we deal with the gluon sector of the theory only, although the inclusion of quarks will be considered later. We first discuss section (3) of Weldon's paper [2]. Consider introducing an action with auxiliary fields $V_0^\mu(Q, x) = V_0^{\mu A}(Q, x)t^A$ and $W_0^\mu(Q, x) = W_0^{\mu A}(Q, x)t^A$ which transform like the field strength under gauge transformations. Here Q is a light-like vector of the form $(1, \hat{\mathbf{q}})$ and the measure $d\Omega(4\pi)^{-1}$ used below averages over all possible $\hat{\mathbf{q}}$ and the label 0 indicates flat space. Weldon [2] defines the action Γ_0 by

$$\begin{aligned} g^2 T^2 X_0(Q; V_0, W_0, A, \eta) &= \int \frac{d\Omega}{4\pi} d^4x 2\text{tr} \left[-\frac{3}{4} m_g^2 W_{0\mu} W_{0\nu} + V_0^\mu Q^\nu F_{\mu\nu} - V_0^\mu Q \cdot D W_{0\mu} \right] \\ \Gamma_0[V_0, W_0, A, \eta] &= g^2 T^2 \int \frac{d\Omega}{4\pi} X_0(Q; V_0, W_0, A, \eta) \end{aligned} \quad (2.3)$$

where $D_\lambda W_\mu = \partial_\lambda W_\mu + ig[A_\lambda, W_\mu]$; $m_g^2 = g^2 T^2 (2N + N_f)/18$ and 'tr' refers to a trace over the colour matrices. $X_0(Q; V_0, W_0, A, \eta)$ is an 'angular integrand' of Γ_0 with

$(Q; V_0, W_0, A, \eta]$ denoting that X_0 is a function of Q and a functional of the various fields.

The Euler-Lagrange equations give

$$\frac{\delta X_0}{\delta V_0} = 0 \Rightarrow (Q \cdot D)W_{0\mu} = Q^\nu F_{\mu\nu} \quad \frac{\delta X_0}{\delta W_0} = 0 \Rightarrow (Q \cdot D)^2 V_{0\mu} = \frac{3}{2}m_g^2 Q^\nu F_{\mu\nu} \quad (2.4)$$

We have not yet specified boundary conditions. These are chosen so that the solutions we obtain from (2.4) for the auxiliary fields, when substituted into (2.3), yield the flat space gluon effective action $\Gamma_0^{HTL}[A, \eta]$. However, we note that $\Gamma_0^{HTL}[A, \eta]$ is not uniquely defined in Minkowski space. [As is well known [6], different $\Gamma_0^{HTL}[A, \eta]$ correspond to different analytic continuations of the uniquely defined Euclidean effective action. This, in turn, corresponds to a choice of writing non-local contributions to $\Gamma_0^{HTL}[A, \eta]$ in a retarded or advanced form]. Thus, the boundary conditions are not uniquely defined by the requirement that we derive a flat space gluon effective action.

Suppose we use homogeneous retarded conditions (i.e. that $W_{0\mu}$, $V_{0\mu} \rightarrow 0$ as $t \rightarrow -\infty$). Then we must take a retarded solution to (2.4) given by

$$W_{0\mu} = (Q \cdot D)_{ret}^{-1}(Q^\nu F_{\mu\nu}) \quad V_{0\mu} = (Q \cdot D)_{ret}^{-2}(Q^\nu F_{\mu\nu}) \quad (2.5)$$

where $(Q \cdot D)_{ret}^{-1}$ is an integral operator whose action on an arbitrary field G^A is defined by

$$\begin{aligned} ((Q \cdot D)_{ret}^{-1}G)^A(x) &= \int_{-\infty}^0 d\theta U^{AB}(x, x + Q\theta) G^B(x + Q\theta) \\ \text{with} \quad U(x, x + Q\theta)^{AB} &= \left(\text{P exp} \int_{\theta}^0 d\theta' [-igQ^\mu \mathcal{A}_\mu(x + Q\theta')] \right)^{AB} \end{aligned} \quad (2.6)$$

The resulting expression on substituting (2.5) into (2.3) gives a retarded form of the flat space gluon effective action

$$\begin{aligned} g^2 T^2 X_0^{HTL}(Q; A, \eta) &= \int d^4x 2\text{tr} \left[-\frac{3}{4}m_g^2 ((Q \cdot D)_{ret}^{-1}Q^\lambda F_{\mu\lambda}) ((Q \cdot D)_{ret}^{-1}Q^\lambda F^\mu{}_\lambda) \right] \\ \Gamma_0^{HTL}[A, \eta] &= g^2 T^2 \int \frac{d\Omega}{4\pi} X_0(Q; A, \eta) \end{aligned} \quad (2.7)$$

Alternatively using homogeneous advanced conditions results in a different, advanced, expression for $\Gamma_0^{HTL}[A, \eta]$ above, where the integral operator $(Q \cdot D)_{ret}^{-1}$ is replaced in (2.7) by $(Q \cdot D)_{adv}^{-1}$ whose action on an arbitrary field G^A is defined by

$$((Q \cdot D)_{adv}^{-1}G)^A(x) = - \int_0^\infty d\theta U^{AB}(x, x + Q\theta) G^B(x + Q\theta) \quad (2.8)$$

No other boundary condition results in a solution of (2.4), which when substituted into (2.3), yields an expression of the form of a possible $\Gamma_0^{HTL}[A, \eta]$.

We impose a constraint on the gluon fields

$$(Q \cdot D)_{ret}^{-1}(Q^\nu F_{\mu\nu}) = (Q \cdot D)_{adv}^{-1}(Q^\nu F_{\mu\nu}) \equiv (Q \cdot D)^{-1}(Q^\nu F_{\mu\nu}) \quad (2.9)$$

We denote the set of gluon fields which satisfy (2.9) by \mathcal{R}_0^{gluon} . For gluon fields in \mathcal{R}_0^{gluon} the flat space gluon hard thermal loop effective action is uniquely defined and the initial choice between advanced and retarded boundary conditions for solutions of (2.4) above becomes redundant.

In the following sections we extend the theory to non-zero κ and encounter several equations for which we have to choose between advanced and retarded boundary conditions, as for (2.4) above. The different choices lead, for arbitrary fields, to effective actions corresponding to different analytic continuations of the uniquely defined Euclidean effective action. For simplicity, we impose constraints on the fields such that, as for solutions of (2.4) above, any choice between retarded or advanced non-local functionals of the fields is actually redundant.

These constraints imply that whenever in the following sections we encounter a non-local expression of the form $\int d\theta' B(x + Q\theta')$, where B is some function of the fields $(\bar{\psi}, \psi, A, e)$, the fields are sufficiently constrained to ensure that

$$\int_{-\infty}^{\theta} d\theta' B(x + Q\theta') = - \int_{\theta}^{\infty} d\theta' B(x + Q\theta') \quad (2.10)$$

Therefore, we effectively restrict the fields so that in the following sections we can always take

$$\int_{-\infty}^{+\infty} d\theta' = 0 \quad (2.11)$$

This imposes a number of constraints on the collection of fields $(\bar{\psi}, \psi, A, e)$ and we denote the set of all such collections by \mathcal{R} . For example, from above we have that the gluon fields are restricted to some subset of \mathcal{R}_0^{gluon} . The effective actions and energy momentum tensors derived below are initially valid only for fields in \mathcal{R} and are uniquely defined on \mathcal{R} . For fields not in \mathcal{R} , the effective actions and energy momentum tensors can be derived by analytic continuation of the expressions valid in \mathcal{R} .

The collections of fields in \mathcal{R} satisfy a great number of constraints, which are outlined below. In the following sections \mathcal{R} is assumed to be non-empty, unless explicitly stated otherwise.

[If \mathcal{R} is an empty set, the results of this paper are still valid. The calculations, as presented below, are invalidated. However, they can easily be repeated but instead of assuming the fields are in \mathcal{R} , we choose some prescription for writing either an advanced or retarded form for any non-local expression encountered below (like solutions of (2.4) above). Different prescriptions lead to effective actions corresponding to different analytic continuations of the uniquely defined Euclidean effective action. The effective action and QGP energy momentum tensor derived using this method are exactly the same as a specific analytic continuation of the effective action and QGP energy momentum tensor derived below. As will be verified below, there is a prescription which yields an effective action for fields not in \mathcal{R} with analyticity properties such that its associated energy momentum tensor is retarded. (The physical significance of retarded currents for the QGP plasma is discussed by Jackiw and Nair [9]).]

2.2 Inclusion of weakly coupling graviton field

Consider a generalisation of the above Minkowskian theory to include interactions of the QGP with a weak graviton field, $\varphi^{\alpha\beta}$. Following [5] and [6] we assume the spacetime to be asymptotically Minkowskian. By considering the symmetries of this theory, we aim to write down an action of local auxiliary fields which, on elimination of the auxiliary fields, gives to $O(\kappa)$ the generating functional Γ^{HTL} of hard thermal loop gluon-graviton interactions, i.e.

$$\Gamma^{HTL}[A, g] = \int \frac{d\Omega}{4\pi} \sum_{m,n} d^4x_1, \dots, d^4x_m d^4y_1, \dots, d^4y_n \quad (2.12)$$

$$\begin{aligned} & A_{\mu_1}(x_1), \dots, A_{\mu_m}(x_m), \varphi^{\alpha_1\beta_1}(y_1), \dots, \varphi^{\alpha_n\beta_n}(y_n) G^{HTL}(x_1, \dots, x_n, y_1, \dots, y_n; Q)_{\alpha_1\beta_1 \dots \alpha_n\beta_n}^{\mu_1 \dots \mu_m} \\ & \equiv g^2 T^2 \int \frac{d\Omega}{4\pi} X^{HTL}(Q; A, g) \end{aligned} \quad (2.13)$$

G^{HTL} refers to the hard thermal 1-loop truncated diagram with n external graviton fields and m external gluon fields. As $\kappa \ll 1$ we take $n = 1$ unless explicitly specified otherwise. Clearly, the definition of X^{HTL} above only specifies X^{HTL} up to a total angular differential. However it is known from hard thermal loop field loop theory [6] that X^{HTL} can be chosen to have the following properties

- A. It is non-local but with non-localities only of the form of products of $(Q \cdot \partial)^{-1}$ operators. Examples of such operators are given by (2.30)
- B. It is homogeneous of degree zero in Q .
- C. It has dimensions of $(\text{energy})^{-2}$.
- D. It is invariant under general coordinate, $SU(N)$ gauge, Weyl scaling (and for spinor theories) local Lorentz transformations which are restricted to tend to the identity at infinity. (A property which all local transformations in this paper are assumed to obey).

When X^{HTL} is referred to in the rest of this paper we implicitly mean the form of X^{HTL} which satisfies the above conditions. [An alternative expression differing by a total angular differential is the basis of the application of Chern-Simons field theory to hard thermal loop phenomena [8]]. In an attempt to produce X^{HTL} at least to $O(\kappa)$ using an auxiliary field lagrangian method, thus enabling a simpler calculation of $T_{\alpha\beta}^{HTL}$, we extend (2.3) to form a general coordinate and Weyl scaling invariant expression. However, we find that is not possible to perform this extension uniquely and that this non-uniqueness has interesting implications, which will be elaborated below.

2.2.1 A Weyl scaling and general coordinate extension

Consider the general coordinate and Weyl scaling invariant extension :

$$g^2 T^2 X_E(Q; V, W, A, g) \equiv \int d^4x (\sqrt{-g} g^{\mu\nu} e^\Lambda) 2\text{tr} \left[-\frac{3}{4} m_g^2 W_\mu W_\nu \right]$$

$$\begin{aligned}
& + V_\mu \dot{y}^\lambda F_{\nu\lambda} - V_\nu [\dot{y}^\alpha \nabla_\alpha W_\mu + W_\alpha \nabla_\mu \dot{y}^\alpha - \dot{y}^\alpha W_\alpha \Lambda_{,\mu}] \\
& + E \left\{ V_\nu [W_\lambda \nabla_\mu \dot{y}^\lambda - \dot{y} \cdot W \partial \Lambda_\mu + g_{\mu\lambda} W \cdot \nabla \dot{y}^\lambda + W_\mu \dot{y} \cdot \partial \Lambda - \dot{y}_\mu W_\alpha \Lambda^{,\alpha}] \right\} \Bigg] \\
\Gamma_E[V, W, A, g] & \equiv g^2 T^2 \int \frac{d\Omega}{4\pi} X_E(Q; V, W, A, g)
\end{aligned} \tag{2.14}$$

where E is an arbitrary constant and ∇_λ refers to a space-time and colour adjoint derivative i.e.

$$\nabla_\lambda W_\mu = \partial_\lambda W_\mu - \Gamma_{\lambda\mu}^\rho W_\rho + ig[A_\lambda, W_\mu]$$

Note that ∇ acting on a colour trivial object, as will occur below, reduces to the standard space time derivative.

In order to define \dot{y}^μ above, we must consider the null geodesic, denoted by $y^\mu(x, \theta)$, which passes through x , is affinely parameterised by θ and satisfies the following conditions

$$y^\mu(x, 0) = x^\mu \quad \dot{y}^\mu(x, \theta) \equiv \frac{dy^\mu(x, \theta)}{d\theta} \rightarrow Q^\mu \quad \text{as } \theta \rightarrow \pm\infty \tag{2.15}$$

The tangent vector of $y^\mu(x, \theta)$ at $\theta = 0$ defines the expression \dot{y}^μ used in (2.14) i.e.

$$\dot{y}^\mu \equiv \dot{y}^\mu(x, \theta) \big|_{\theta=0} \equiv \frac{dy^\mu(x, \theta)}{d\theta} \bigg|_{\theta=0} \tag{2.16}$$

Note that \dot{y}^μ generalises Q^μ . Using the null geodesic equation

$$\ddot{y}(x, \theta) = -\Gamma_{\mu\nu}^\lambda(y(x, \theta)) \dot{y}^\mu(x, \theta) \dot{y}^\nu(x, \theta) \tag{2.17}$$

and the conditions (2.15) we can expand $\dot{y}^\mu(x, \theta)$ in powers in of κ . Using (2.11), we have to first order in κ

$$\begin{aligned}
\dot{y}^\lambda(x, \theta) &= Q^\lambda - \kappa \int_\theta^\infty d\theta' \gamma^\lambda(x + Q\theta') + O(\kappa^2) \\
&= Q^\lambda + \kappa \int_{-\infty}^\theta d\theta' \gamma^\lambda(x + Q\theta') + O(\kappa^2) \\
\text{where } \gamma^\lambda &= -Q_\mu (Q \cdot \partial) \phi^{\lambda\mu} + \frac{1}{2} Q_\mu Q_\nu \partial^\lambda \phi^{\mu\nu}
\end{aligned} \tag{2.18}$$

Note that $\int_{-\infty}^\theta d\theta' \gamma^\lambda(x + Q\theta') = -\int_\theta^\infty d\theta' \gamma^\lambda(x + Q\theta')$ gives a constraint on the vierbien associated with the metric. This constraint therefore must be satisfied by any vierbien belonging to \mathcal{R} . Similar restrictions arise at higher orders in κ . Without such restrictions we can choose $\dot{y}^\lambda(x, \theta)$ such that $(\dot{y}^\lambda(x, \theta) - Q^\lambda)$ satisfies either homogeneous retarded or homogeneous advanced boundary conditions, but not both.

Λ is a dimensionless scalar, homogeneous of degree zero in Q , which vanishes when $g = \eta$ with the Weyl scaling property that $\Lambda \rightarrow \Lambda + 2\sigma$.

[Recall that a Weyl scaling is given by $g^{\alpha\beta} \rightarrow e^{2\sigma} g^{\alpha\beta}$, $g_{\alpha\beta} \rightarrow e^{-2\sigma} g_{\alpha\beta}$, $A_\mu \rightarrow A_\mu$, $\psi \rightarrow e^{\frac{3\sigma}{2}} \psi$ where for this paper we assume $\sigma \rightarrow 0$ as $x \rightarrow \infty$].

We can give a simple closed form expression for Λ in terms of $\dot{y}^\mu(x, \theta)$ defined above. First recall that a null geodesic, although invariant under a Weyl scaling, is reparameterised so that its affine parameter θ transforms as $d\theta \rightarrow e^{-2\sigma} d\theta$. Thus

$$\dot{y}^\mu(x, \theta) \equiv \frac{dy^\mu(x, \theta)}{d\theta} \rightarrow e^{2\sigma(y(x, \theta))} \dot{y}^\mu(x, \theta) \quad (2.19)$$

on a Weyl scaling .

Taking into account the well known (see eg. [11]) Weyl scaling properties of the Christoffel symbol, it is straightforward to show that

$$(\nabla_\alpha \dot{y}^\alpha) \rightarrow e^{2\sigma} (\nabla_\alpha \dot{y}^\alpha - 2(\dot{y}^\alpha \partial_\alpha \sigma)) \quad (2.20)$$

on Weyl scaling and hence Λ can be defined by the solution of the equation

$$(\dot{y} \cdot \partial) \Lambda = -(\nabla_\alpha \dot{y}^\alpha) \quad (2.21)$$

Again using (2.11), we have

$$\Lambda(y(x, \theta)) = + \int_\theta^\infty d\theta' \nabla_\nu \dot{y}^\nu(y(x, \theta')) = - \int_{-\infty}^\theta d\theta' \nabla_\nu \dot{y}^\nu(y(x, \theta')) \quad (2.22)$$

Again the equality of advanced and retarded versions of Λ requires a constraint on the vierbiens fields belonging to \mathcal{R} . We have

$$\Lambda(x) \equiv \Lambda(y(x, 0)) \rightarrow \Lambda(y(x, 0)) + 2\sigma(x) = \Lambda(x) + 2\sigma(x) \quad (2.23)$$

on Weyl scaling as required. It is also straightforward to check this definition is consistent with all the other properties required of Λ .

Note however Λ is not uniquely defined at $O(\kappa^2)$ and the consequences of this are discussed in Appendix C. Below, we use Λ as given in (2.22). However, as we are working at $O(\kappa)$ throughout this paper (except briefly in Appendix C), all of the following discussion is independent of the second order ambiguity in Λ .

$X_E(Q; V, W, A, g)$ above has been constructed to be Weyl scaling invariant if V and W are Weyl scaling invariant. [Note that although the Weyl scaling properties imposed on W and V appear to be arbitrary , this apparent freedom doesn't actually affect the Weyl scaling and general coordinate invariant extension of the flat space action at $O(\kappa)$, as demonstrated in Appendix B].

To insure Weyl scaling invariance of X_E it is not sufficient to generalise $Q \cdot DW_\mu$ to $\dot{y} \cdot \nabla W_\mu$ due to the awkward Weyl scaling properties of the Christoffel symbols present in ∇ and hence extra terms are added so that

$$Q \cdot DW_\mu \rightarrow \dot{y}^\alpha \nabla_\alpha W_\mu + W_\alpha \nabla_\mu \dot{y}^\alpha - \dot{y}^\alpha W_\alpha \Lambda_{,\mu} = \dot{y}^\alpha \partial_\alpha W_\mu + W_\nu \partial_\mu \dot{y}^\nu - \dot{y}^\alpha W_\alpha \Lambda_{,\mu}$$

Thus the Christoffel symbol of $W_\nu \nabla_\mu \dot{y}^\nu$ cancels the Christoffel symbol of $\dot{y}^\alpha \nabla_\alpha W_\mu$ in a covariant manner. Using (2.19) and (2.23) is straightforward to verify that the

above expression is simply multiplied by $e^{2\sigma}$ on Weyl scaling. This ad-hoc fixing of Weyl scaling is not unique and an alternative is given by

$$Q \cdot DW_\mu = \eta_{\mu\nu} Q \cdot DW^\nu \rightarrow g_{\mu\nu} [\dot{y}^\alpha \nabla_\alpha W^\nu - W^\mu \nabla_\mu \dot{y}^\nu - W^\nu \dot{y} \cdot \partial \Lambda + \dot{y}^\nu W \cdot \partial \Lambda]$$

The difference between these two possibilities gives the coefficient of E in (2.14) above and thus E is an arbitrary parameter, reflecting the ambiguity present in extending Weldon's action. Of course, we don't know yet for what value of E (if any) (2.14) will yield X^{HTL} (to $O(\kappa)$ at least) on elimination of the auxiliary fields. Further ambiguities exist at $O(\kappa^2)$ and are discussed in Appendix C.

2.2.2 Elimination of the auxiliary fields

Let $X_E(Q; A, g]$ and $\Gamma_E[A, g]$ denote the angular integrand and the action formed by the elimination of the auxiliary fields from $X_E(Q; V, W, A, g]$ and $\Gamma_E[V, W, A, g]$ using the Euler-Lagrange equations. Thus,

$$\begin{aligned} g^2 T^2 X_E(Q; A, g] &= \int d^4x (\sqrt{-g} g^{\mu\nu} e^\Lambda) 2\text{tr}(-\tfrac{3}{4} m_g^2 W_\mu W_\nu) \\ \Gamma_E[A, g] &= g^2 T^2 \int \frac{d\Omega}{4\pi} X_E(Q; A, g] \end{aligned} \quad (2.24)$$

where W is a solution of the Euler Lagrange equations

$$\begin{aligned} \dot{y}^\alpha \nabla_\alpha W_\mu &= \dot{y}^\nu F_{\mu\nu} + W_\alpha (\Delta^E)^\alpha_\mu \\ (\Delta^E)^\alpha_\mu &= \dot{y}^\alpha \Lambda_{,\mu} - \nabla_\mu \dot{y}^\alpha + E [\nabla_\mu \dot{y}^\alpha - \dot{y}^\alpha \Lambda_{,\mu} + \tfrac{1}{2} g_\mu^\alpha \dot{y} \cdot \partial \Lambda + (\alpha \leftrightarrow \mu)] \end{aligned} \quad (2.25)$$

Clearly any solution W_μ to the above equation is covariant under general coordinate transformations, transforms like the field strength under gauge transformations and has dimensions L^{-1} . A solution of (2.25) can be obtained as a perturbative expansion in κ . Writing

$$\begin{aligned} W_\mu &= \sum_{n=0} \kappa^n W_{n\mu}, \quad (\dot{y} \cdot \nabla) = Q \cdot D + \sum_{n=1} \kappa^n (\dot{y} \cdot \nabla)_n, \quad \dot{y}^\nu = Q^\nu + \sum_{n=1} \kappa^n \dot{y}_n^\nu \\ (\Delta^E)^\alpha_\mu &= \sum_{n=1} \kappa^n (\Delta_n^E)^\alpha_\mu \end{aligned} \quad (2.26)$$

we have

$$W_{0\mu} = (Q \cdot D)^{-1} (Q^\nu F_{\mu\nu}) \quad (2.27)$$

$$W_{n\mu} = (Q \cdot D)^{-1} \{ \dot{y}_n^\nu F_{\mu\nu} + \sum_{p=0}^{n-1} [-(\dot{y} \cdot \nabla)_{n-p} W_{p\mu} + (\Delta_{n-p}^E)^\alpha_\mu W_{p\alpha}] \} \quad (2.28)$$

Thus $W_{0\mu}$ agrees with the flat space auxiliary field in section (2.1). We impose sufficient restrictions on the gluon and vierbien fields in \mathcal{R} to ensure that the choice of a retarded or advanced inverse in (2.28) is redundant.

By expanding $(Q \cdot D)^{-1}$ in powers of g , it is straightforward to verify that each $W_{n\mu}$ and hence W_μ consists of non-localities of the form of products of $(Q \cdot \partial)^{-1}$. Applying $Q \cdot \frac{\partial}{\partial Q}$

to both sides of equation (2.28) we see (by inducting on n) that each $W_{n\mu}$ is homogeneous of degree zero in Q and hence so is W_μ .

W is also Weyl invariant as this is imposed on W when making the extension from flat space. Note that any solution of (2.28) must be Weyl invariant as it can be shown, by induction on n , that each $W_{n\mu}$ is Weyl invariant.

Hence W_μ has the correct properties to ensure $X_E(Q; A, g]$ obeys (A) \rightarrow (D) as given in section (2.2). This fact, together with the observation that in the limit $g \rightarrow \eta$, X_E reduces to X_0 , is interesting and contrary to the previous assertion that a functional satisfying conditions (A) \rightarrow (D) and with the correct flatspace terms would be in fact equal to $X^{HTL}(Q; A, g]$ at higher orders in κ [6, 7]. This is considered in more detail in section (A.2) of Appendix A.

2.2.3 The relation between $X_E(Q; A, g]$ and $X^{HTL}(Q; A, g]$

We show in Appendix A that the following holds: given any expression which

- i. Satisfies conditions (A) \rightarrow (D) above.
- ii. Reduces to $X_0^{HTL}(Q; A, \eta]$ when $g \equiv \eta$.
- iii. Agrees with $X^{HTL}(Q; A, g]$ at $O(g^2\kappa)$.

must, at least to $O(\kappa)$, be equal to $X^{HTL}(Q; A, g]$ for all orders in g .

Now consider $X_E[A, g]$ above. We know it satisfies conditions (i) and (ii) above. We will show that for $E = 0$, it also agrees with X^{HTL} at $O(g^2\kappa)$. First, we calculate $S_{\alpha\beta}^E$ defined by

$$S_{\alpha\beta}^E \equiv 2g^2 T^2 \lim_{g \rightarrow \eta} \left(\frac{\delta X_E(Q; A, g]}{\delta g^{\alpha\beta}} \right)$$

(so that the angular integral of the above is equal to the energy momentum tensor associated with the action Γ_E). Then we explicitly verify that for $E = 0$, $S_{\alpha\beta}^{E=0}$ is equal to the $O(g^2)$ contribution to the angular integrand of $T_{\alpha\beta}^{HTL}$ which is consistent with conditions (A) \rightarrow (D) (as calculated using thermal field theory in [6]). This shows equality between X_E and X^{HTL} at $O(g^2\kappa)$ which explicitly confirms condition (iii). Hence we deduce that $X_{E=0}(Q; A, g] = X^{HTL}(Q; A, g]$ at least to $O(\kappa)$ and hence that $\int d\Omega (4\pi)^{-1} S_{\alpha\beta}^{E=0}$ is equal to the gluon contribution of the retarded QGP energy momentum tensor, $T_{\alpha\beta}^{HTL}$, to all orders in g .

2.3 Calculation of $S_{\alpha\beta}^E$ and the QGP energy momentum tensor

The calculation of $S_{\alpha\beta}^E$ is simplified by noting that for when V, W are considered as functionals of the physical fields ie. $V = V(Q; A, g]$, $W = W(Q; A, g]$ we have

$$\begin{aligned} \frac{\delta X_E(Q; V, W, A, g]}{\delta g^{\alpha\beta}} \Big|_{A, V, W} &= \frac{\delta X_E(Q; V, W, A, g]}{\delta g^{\alpha\beta}} \Big|_A - \frac{\delta X_E(Q; V, W, A, g]}{\delta V_\mu} \Big|_{A, g, W} \frac{\delta V_\mu(Q; A, g]}{\delta g^{\alpha\beta}} \\ &- \frac{\delta X_E(Q; V, W, A, g]}{\delta W_\mu} \Big|_{A, g, V} \frac{\delta W_\mu(Q; A, g]}{\delta g^{\alpha\beta}} \end{aligned}$$

but with $g = \eta$, $V = V_0$, and $W = W_0$ (see (2.4)) we have $\frac{\delta X_E(Q; V, W, A, g]}{\delta V_\mu}$ and $\frac{\delta X_E(Q; V, W, A, g]}{\delta W_\mu}$ zero and hence

$$S_{\alpha\beta}^E = 2g^2 T^2 \lim_{g \rightarrow \eta} \left(\frac{\delta X_E(Q; V = V_0[A, g], W = W_0[A, g], A, g]}{\delta g^{\alpha\beta}} \Big|_{A, V, W \text{ const}} \right) \quad (2.29)$$

Note that calculating $S_{\alpha\beta}^E$ only requires the flat space equations for the auxiliary fields and thus that $X_E(Q; A, g]$ at $O(\kappa)$ depends only on the flat space auxiliary fields. Performing the calculation it is useful to note the following

$$\int d^4x G(x) ((Q \cdot \partial)_{ret}^{-1} F)(x) = - \int d^4x ((Q \cdot \partial)_{adv}^{-1} G)(x) F(x) \quad (2.30)$$

$$\text{where } ((Q \cdot \partial)_{ret}^{-1} F)(x) = \int_{-\infty}^0 d\theta F(x + Q\theta) \quad ((Q \cdot \partial)_{adv}^{-1} F)(x) = - \int_0^\infty d\theta F(x + Q\theta)$$

The labels ‘ret’ and ‘adv’ can be dropped if which $\int d\theta' F(x + Q\theta')$ and $\int d\theta' G(x + Q\theta')$ are non-local functionals of the fields for which (2.11) holds. We impose sufficient restrictions for fields in \mathcal{R} to ensure that all non-local functionals encountered below satisfy (2.11).

Hence from the definitions of \dot{y} and Λ , we have for any F_λ and F

$$\begin{aligned} \int dx \dot{y}^\lambda F_\lambda &= \int dx \phi^{\alpha\beta} \frac{1}{2} [Q_{\{\alpha} \eta_{\beta\}}^\lambda - Q_\alpha Q_\beta (Q \cdot \partial)^{-1} \partial^\lambda] F_\lambda \\ \int dx \Lambda F &= \int dx \phi^{\alpha\beta} \frac{1}{2} [Q_\alpha Q_\beta \square (Q \cdot \partial)^{-2} + \eta_{\alpha\beta} - Q_{\{\alpha} \partial_{\beta\}} (Q \cdot \partial)^{-1}] F \\ \text{where } A_{\{\alpha} B_{\beta\}} &= A_\alpha B_\beta + A_\beta B_\alpha \end{aligned} \quad (2.31)$$

On performing the calculation we find

$$\begin{aligned} S_{\alpha\beta}^E &= 4\text{tr} \left(\frac{1}{2} [Q_\alpha Q_\beta \square (Q \cdot \partial)^{-2} - Q_{\{\alpha} \partial_{\beta\}} (Q \cdot \partial)^{-1}] \left[-\frac{3}{4} m_g^2 W_0^\mu W_{0\mu} \right] - \frac{3}{4} m_g^2 W_{0\alpha} W_{0\beta} \right. \\ &\quad + \frac{1}{2} [Q_{\{\alpha} \eta_{\beta\}}^\lambda - Q_\alpha Q_\beta (Q \cdot \partial)^{-1} \partial^\lambda] [V_0^\mu F_{\mu\lambda} - V_{0\mu} D_\lambda W_0^\mu + \partial_\mu (W_{0\lambda} V_0^\mu)] \\ &\quad + \frac{1}{2} E \left\{ [Q \cdot \partial (W_{0\{\beta} V_{0\alpha\}})] - [Q_{\{\alpha} \eta_{\beta\}}^\lambda - Q_\alpha Q_\beta (Q \cdot \partial)^{-1} \partial^\lambda] [\partial_\mu (W_{0\lambda} V_0^\mu + V_{0\lambda} W_0^\mu)] \right. \\ &\quad \left. \left. - [Q_\alpha Q_\beta \square (Q \cdot \partial)^{-2} + \eta_{\alpha\beta} - Q_{\{\alpha} \partial_{\beta\}} (Q \cdot \partial)^{-1}] [Q \cdot \partial (V_{0\mu} W_0^\mu)] \right\} \right) \end{aligned} \quad (2.32)$$

Note $V_{0\mu}$ and $W_{0\mu}$ are given by (2.4) and we have used $Q^\mu V_{0\mu} = Q^\mu W_{0\mu} = 0$ in deriving the above. The coefficient of E in (2.32) is discussed further in section A.2 of Appendix A.

Setting $E=0$ we find $S_{\alpha\beta}^{E=0}$ gives equation (6) of [6] in momentum space at $O(g^2)$ thus showing, given the result of Appendix A, that $X_{E=0}(Q; A, g] = X^{HTL}(Q; A, g]$ at $O(\kappa)$ for all orders in g . Thus $S_{\alpha\beta}^{E=0}$ is the angular integrand of the gluon contribution to the QGP energy momentum tensor $T_{\alpha\beta}^{HTL}$ and hence

$$T_{\alpha\beta}^{HTL} = \int \frac{d\Omega}{4\pi} S_{\alpha\beta}^{E=0} \quad (2.33)$$

The calculations deriving this are valid for gluon fields in \mathcal{R} . We can analytically continue from \mathcal{R} to consider this expression for gluon fields not in \mathcal{R} . A choice of analytic

continuation where all fields are retarded is of particular physical significance [9]. Despite using a local auxiliary field angular integrand, $T_{\alpha\beta}^{HTL}$ is not local even with respect to the auxiliary fields. This is because \dot{y}^μ and Λ have a non-local dependence on the graviton fields.

An interesting but difficult question is whether or not $X_{E=0}(Q; A, g]$ is correct at $O(\kappa^2)$ and some of the difficulties that arise are discussed in Appendix C.

[We again consider the problem of choosing a prescription which yields a retarded energy momentum tensor. As discussed in section (2.1), such an approach is required if we drop the assumption that \mathcal{R} is non-empty. The prescription is to choose the advanced expressions for \dot{y} and Λ in (2.18) and (2.22), while choosing retarded expressions for all other possibilities. With this choice of prescription we see that, on use of (2.30), (2.31) is still valid on replacing $(Q \cdot \partial)^{-1}$ by $(Q \cdot \partial)_{ret}^{-1}$. This would therefore lead to retarded energy momentum tensor, automatically valid for fields not in \mathcal{R}].

3 Single auxiliary field model

3.1 Flat space auxiliary fields

The same analysis is applied to a model with only one auxiliary field $m_0^\mu(Q, x) = m_0^{\mu A}(Q, x)t^A$ which again transforms like the field strength on gauge transformations.

Consider the following angular integrand $X_0(Q; m_0, A, \eta]$ and action $\Gamma_0[m_0, A, \eta]$, as flat space precursors:

$$\begin{aligned} g^2 T^2 X_0(Q; m_0, A, \eta] &= \int d^4x (4b) \text{tr}[m_0^\lambda Q^\nu F_{\lambda\nu} + \frac{1}{2}(Q \cdot D m_{0\nu})(Q \cdot D m_0^\nu)] \quad (3.34) \\ \Gamma_0[m_0, A, \eta] &\equiv g^2 T^2 \int \frac{d\Omega}{4\pi} X_0(Q; m_0, A, \eta] \end{aligned}$$

On taking $b = \frac{3}{4}m_g^2$ and using (2.11) together with the Euler-Lagrange equation for m , the above reduce to the non-local flat space hard thermal loop angular integrand and action, i.e.

$$\begin{aligned} \frac{\delta X}{\delta m} = 0 &\Rightarrow m_{0\mu} = (Q \cdot D)^{-2} Q^\nu F_{\mu\nu} \Rightarrow \quad (3.35) \\ g^2 T^2 X_0^{HTL}(Q; A, \eta] &= - \int d^4x \quad 2\text{tr}[\frac{3}{4}m_g^2[(Q \cdot D)^{-1} Q^\lambda F_{\mu\lambda}] [(Q \cdot D)^{-1} Q^\lambda F^\mu{}_\lambda]] \\ \Gamma_0[A, \eta] &\equiv g^2 T^2 \int \frac{d\Omega}{4\pi} X_0(Q; A, \eta] \end{aligned}$$

Note that above we have used

$$\int d^4x (Q^\nu F^\mu{}_\nu) [(Q \cdot D)^{-2} (Q^\nu F_{\mu\nu})] = - \int d^4x [(Q \cdot D)^{-1} (Q^\nu F_{\mu\nu})] [(Q \cdot D)^{-1} (Q^\nu F^\mu{}_\nu)] \quad (3.36)$$

which is valid for fields in \mathcal{R} .

3.2 Inclusion of Weakly coupling graviton field

Again, on generalising to curved space and imposing Weyl invariance, we hope to obtain $X^{HTL}[A, g]$ on the elimination of the auxiliary fields. As in section 2 we find there are

ambiguities. Consider the generalisation where E' is an arbitrary constant:

$$g^2 T^2 X_{E'} = \int d^4 x (\sqrt{-g} e^\Lambda) 2\text{tr} \left[2b(\eta_\rho^\nu m^\rho \dot{y}^\lambda F_{\nu\lambda}) \right. \\ \left. + b g_{\mu\nu} (B^\mu B^\nu + E' B^\mu C^\nu + \frac{1}{2} E'^2 C^\mu C^\nu) \right] \quad (3.37)$$

$$\text{where } B^\mu = (\dot{y}^\alpha \nabla_\alpha m^\mu - m^\lambda \nabla_\lambda \dot{y}^\mu + \dot{y}^\mu m^\lambda \partial_\lambda \Lambda) \quad (3.38)$$

$$C^\nu = (m_\rho \nabla^\nu \dot{y}^\rho - m \cdot \dot{y} \Lambda_{,\nu} + m^\nu \dot{y} \cdot \nabla \Lambda + m \cdot \nabla \dot{y}^\nu - \dot{y}^\nu m \cdot \nabla \Lambda) \quad (3.39)$$

with $m^\mu \rightarrow m^\mu$ on Weyl scaling. Again the choice of the Weyl scaling properties doesn't affect $X_{E'}(Q; m, A, g)$ at $O(\kappa)$ (see Appendix B) and the choice above is for simplicity.

The coefficients of E' & E'^2 correspond to the fact that the extension to a general coordinate and Weyl scaling invariant action is not unique and they are due to taking different combinations of the following two possibilities for the generalisation of $Q \cdot Dm^\mu$ (both of which are simply multiplied by $e^{2\sigma}$ on Weyl scaling).

$$Q \cdot Dm^\mu \rightarrow \dot{y}^\alpha \nabla_\alpha m^\mu - m^\alpha \nabla_\alpha \dot{y}^\mu + \dot{y}^\mu m^\alpha \partial_\alpha \Lambda \\ Q \cdot Dm^\mu = \eta^{\mu\nu} Q \cdot Dm_\nu \rightarrow g^{\mu\nu} [\dot{y}^\alpha \nabla_\alpha m_\nu + m_\rho \nabla_\nu \dot{y}^\rho - m^\alpha \dot{y}_\alpha \Lambda_{,\nu} + m_\nu \dot{y}^\alpha \partial_\alpha \Lambda]$$

The Euler-Lagrange equations for m^μ are

$$\dot{y} \cdot \nabla (B_\alpha + E' C_\alpha) = \dot{y}^\lambda F_{\alpha\lambda} + (B_\nu + E' C_\nu) (\Delta^{E'})^\alpha{}_\mu \quad (3.40)$$

where $(\Delta^{E'})^\alpha{}_\mu$ is given by (2.25) (for $E \rightarrow E'$). Thus by (2.25) we see that W_μ solves (3.40) and hence we have

$$\dot{y}^\alpha \nabla_\alpha m^\mu = g^{\mu\nu} W_\nu - (\Delta^{E'})^\mu{}_\nu m^\nu \quad (3.41)$$

This is solved in exactly the same way as (2.25) (after raising indices and replacing $\dot{y}^\nu F^\mu{}_\nu$ by $g^{\mu\nu} W_\nu$). Clearly any solution m^μ to the above equation is covariant under general coordinate transformations, transforms like the field strength under gauge transformations and has dimensions L^0 . By expanding in powers of g and κ it is straightforward to show that its non-localities are only of the form of products of $(Q \cdot \partial)^{-1}$. Writing $m^\mu = \Sigma_{n=0} \kappa^n m_n^\mu$, it is also straightforward to prove [by induction on n exactly as in section (2.2.2)] that m^μ is homogeneous in Q of degree minus one and that its imposed Weyl invariance is consistent with the Euler Lagrange equations.

Defining $X_{E'}^m(Q; A, g)$ by (3.37) *at the solution of (3.40)* we see that m^μ has the correct properties to ensure that $X_{E'}^m(Q; A, g)$ satisfies condition (i). Section (3.1) shows that it satisfies condition (ii). Thus we see that $X_{E'}^m(Q; A, g)$ is a counterexample to the assertion that a functional satisfying conditions (A) \rightarrow (D) and with the correct flatspace terms would be in fact equal to $X^{HTL}(Q; A, g)$ at higher orders in κ [6, 7]. [In fact, as we will show below $X_{E'}^m(Q; A, g) = X_E(Q; A, g)$ at $O(\kappa)$ for $E = E'$].

In addition condition (iii) is satisfied for $E' = 0$. We show this by simply calculating

$${}^m S_{\alpha\beta}^{E'} \equiv 2g^2 T^2 \lim_{g \rightarrow \eta} \frac{\delta X_{E'}^m(Q; A, g)}{\delta g^{\alpha\beta}} \quad (3.42)$$

and verifying that ${}^m S_{\alpha\beta}^{E'=0} = S_{\alpha\beta}^{E=0}$ (which is equal to the angular integrand of $T_{\alpha\beta}^{HTL}$ which satisfies (A) \rightarrow (D) by the results of section 2).

Using the techniques of section (2.3) we find ${}^m S_{\alpha\beta}^{E'}$ is a function of the gluon fields given by

$$\begin{aligned} {}^m S_{\alpha\beta}^{E'} = & 4b\text{tr}\left\{\frac{1}{2}[Q_\alpha Q_\beta \square(Q \cdot \partial)^{-2} - Q_{\{\alpha} \partial_{\beta\}}(Q \cdot \partial)^{-1}][m_{0\mu} f^\mu + \frac{1}{2}Q \cdot Dm_{0\mu} Q \cdot Dm_0^\mu] \right. \\ & + \frac{1}{2}[Q_{\{\alpha} \eta_{\beta\}}^\lambda - Q_\alpha Q_\beta (Q \cdot \partial)^{-1} \partial^\lambda][m_{0\mu} F^\mu{}_\lambda + (D_\lambda m_{0\mu})Q \cdot Dm_0^\mu \\ & \quad \left. + \partial^\mu(m_{0\mu} Q \cdot Dm_{0\lambda})] - \frac{1}{2}(Q \cdot Dm_{0\alpha} Q \cdot Dm_{0\beta}) \right. \\ & + \frac{1}{2}E' \left[-[Q_{\{\alpha} \eta_{\beta\}}^\lambda - Q_\alpha Q_\beta (Q \cdot \partial)^{-1} \partial^\lambda][\partial_\mu(Q \cdot m_{0\lambda} m_0^\mu + m_{0\lambda} Q \cdot m_0^\mu)] \right. \\ & - [Q_\alpha Q_\beta \square(Q \cdot \partial)^{-2} + \eta_{\alpha\beta} - Q_{\{\alpha} \partial_{\beta\}}(Q \cdot \partial)^{-1}][Q \cdot \partial(m_{0\mu} Q \cdot m_0^\mu)] \\ & \left. + Q \cdot \partial(Q \cdot m_{0\{\beta} m_{0\alpha\}})] \right\} \end{aligned} \quad (3.43)$$

where $f^\mu = Q^\rho F^\mu{}_\rho$ and $m_{0\mu}$ is given by (3.40). Note that we have used $Q^\mu m_{0\mu} = 0$ in deriving the above and that the coefficient of E' in ${}^m S_{\alpha\beta}^{E'}$ above is the same as the coefficient of E in (2.32) which is consistent with uniqueness results stated in section (2) (and derived in Appendix A).

Setting $E' = 0$ in (3.43), applying the identity

$$(D_\lambda m_{0\mu})Q \cdot Dm_{0\mu} = \partial_\lambda\{(Q \cdot \partial)^{-1}(Q \cdot \partial)[(Q \cdot Dm_0^\mu)m_{0\mu}]\} - m_0^\mu D_\lambda(Q \cdot Dm_{0\mu})$$

in the second line above and noting that $W_{0\mu} = (Q \cdot D)m_{0\mu}$ and $V_{0\mu} = 2b Q \cdot Dm_{0\mu}$ it is straightforward to see that ${}^m S_{\alpha\beta}^{E'=0}$ is equal to $S_{\alpha\beta}^{E=0}$ derived earlier.

Hence $X_{E'=0}^m(Q; A, g)$ satisfies conditions (i) \rightarrow (iii). Indeed, we see from the equivalence of $S_{\alpha\beta}^E$ and ${}^m S_{\alpha\beta}^{E'}$ (for $E = E'$) that the actions presented in section (2) and this section are equivalent at $O(\kappa)$ and hence that the results of section (2) could be derived, ab initio, with a model using only one auxiliary field.

Again, considering agreement of these actions and the gluon graviton hard thermal loop effective action at $O(\kappa^2)$ is a more difficult problem.

4 Quark Sector

4.1 Flat space auxiliary fields

Again we perform the analysis of section 2 only now for the quark sector of the theory, with the aim of deriving an expression for

$$\lim_{e \rightarrow \eta} \left[e^a \frac{\delta \Gamma^{HTL}(\text{quark})[\bar{\psi}, \psi, A, g]}{\delta e^{a\beta}} \right] \quad (4.44)$$

where e is the vierbien field.

We start from a flat space angular integrand $X_{\text{quark}(0)}$ and action $\Gamma_{\text{quark}(0)}$ with a dependence on the auxiliary fields $\chi_0(Q, x)$ and $\Psi_0(Q, x)$, the latter of which acts as a lagrange multiplier (cf. $V(Q, x)$ in section 2).

$$\begin{aligned} g^2 T^2 X_{\text{quark}(0)} &= \int d^4 x a (\bar{\psi} \chi_0 + \bar{\Psi}_0 (\not{Q} \psi + iQ \cdot \tilde{D} \chi_0) + \text{h.c.}) \\ \Gamma_{\text{quark}(0)} &= g^2 T^2 \int \frac{d\Omega}{4\pi} X_{\text{quark}(0)} \end{aligned}$$

where $a = g^2 T^2 (N^2 - 1)/(16N)$ and \tilde{D}^λ is the covariant derivative acting on fields in the fundamental representation ie. $\tilde{D}^\lambda \psi = \partial^\lambda + ig A^\lambda \psi$. The Euler-Lagrange equations give

$$(Q \cdot \tilde{D})\chi_0 = i\mathcal{Q}\psi \quad \text{and} \quad (Q \cdot \tilde{D})\Psi_0 = i\psi \quad (4.45)$$

Again, we impose sufficient restrictions on the fields belonging to \mathcal{R} to ensure that (2.11) holds for the non-local functionals of the fields which we encounter in the quark sector. Thus we have

$$\chi_0 = i(Q \cdot \tilde{D})^{-1} \mathcal{Q}\psi \quad \Psi_0 = i(Q \cdot \tilde{D})^{-1} \psi \quad (4.46)$$

$$\text{with} \quad (Q \cdot \tilde{D})_{ret}^{-1} \psi = (Q \cdot \tilde{D})_{adv}^{-1} \psi \equiv (Q \cdot \tilde{D})^{-1} \psi \quad (4.47)$$

where $(Q \cdot \tilde{D})_{ret}^{-1}$ and $(Q \cdot \tilde{D})_{adv}^{-1}$ are integral operators similar to (2.6) and given by

$$((Q \cdot \tilde{D})_{ret}^{-1} \psi)^A(x) = \int_{-\infty}^0 d\theta U(x, x + Q\theta) \psi(x + Q\theta) \quad (4.48)$$

$$((Q \cdot \tilde{D})_{adv}^{-1} \psi)^A(x) = - \int_0^\infty d\theta U(x, x + Q\theta) \psi(x + Q\theta)$$

$$\text{with} \quad U(x, x + Q\theta) = \left(\text{P exp} \int_\theta^0 d\theta' [-ig Q^\mu A_\mu(x + Q\theta')] \right)$$

Thus we have on elimination of the auxiliary fields and using $[\mathcal{Q}, Q \cdot \tilde{D}] = 0$,

$$X_{(0)quark} = ia\bar{\psi} \mathcal{Q} (Q \cdot \tilde{D})^{-1} \psi + \text{h. conj.} \quad (4.49)$$

This is the quark flatspace hard thermal loop angular integrand denoted $X_{quark(0)}^{HTL}$ and the above calculation is valid for fields in \mathcal{R} . We can consider (4.49) for fields not in \mathcal{R} by analytic continuation.

4.2 Inclusion of weakly coupling graviton field to quark sector

Again, we consider generalising to curved space and imposing Weyl invariance noting that the Weyl transformation properties of the auxiliary fields once more do not affect the action at $O(\kappa)$ (see Appendix B). Thus Ψ and χ can be considered, without loss of generality, as Weyl invariants. Unlike sections 2 and 3 we find no ambiguities on generalising the angular integrand and action. This is consistent with the fact that quark sector analogues of the gluon conditions (i) and (ii) (given by (i)' and (ii)' in section (A.3) of Appendix A) are sufficient, without an analogue of the gluon condition (iii) above, to determine $X^{HTL(quark)}$. (See section A.3 of Appendix A for further details). This makes the quark analysis relatively straightforward. $\Gamma^{HTL(quark)}$ and $X^{HTL(quark)}$ are defined analogously to their gluon counterparts ie.

$$\begin{aligned} \Gamma^{HTL(quark)} &= \int \frac{d\Omega}{4\pi} d^4 z_1 d^4 z_2 \sum_{m,n} d^4 x_1, \dots, d^4 x_m d^4 y_1, \dots, d^4 y_n A_{\mu_1}(x_1), \dots, A_{\mu_m}(x_m) \\ &\quad \varphi^{\alpha_1 \beta_1}(y_1), \dots, \varphi^{\alpha_n \beta_n}(y_n) \bar{\psi}(z_1) G^{HTL(quark)}(x_1, \dots, x_n, y_1, \dots, y_n; Q)_{\alpha_1 \beta_1 \dots \alpha_n \beta_n}^{\mu_1 \dots \mu_m} \psi(z_2) \\ &\equiv g^2 T^2 \int \frac{d\Omega}{4\pi} X^{HTL(quark)}(Q; \bar{\psi}, \psi, A, g) \end{aligned} \quad (4.50)$$

where $G^{HTL(quark)}$ refers to the hard thermal 1-loop truncated diagram with a quark anti-quark pair of external fields, n external graviton fields and m external gluon fields. As before we take $n = 1$ and assume that $X^{HTL(quark)}$ satisfies conditions (A) \rightarrow (D) given above. Consider the Weyl scaling and general coordinate invariant generalisation of the auxiliary field angular integrand in section (4.1) :

$$g^2 T^2 X_{quark} = \int d^4x \sqrt{-g} a (\bar{\psi} e^{\frac{5\Lambda}{4}} \chi + \bar{\Psi} (e^{\frac{5\Lambda}{4}} (e^{-\frac{\Lambda}{2}} \dot{\not{y}}) \psi + i e^\Lambda \dot{y} \cdot \tilde{\nabla} \chi) + \text{h. conj.}) \quad (4.51)$$

$$\Gamma_{quark} = \int \frac{d\Omega}{4\pi} X_{quark}$$

where

$$(\dot{y} \cdot \tilde{\nabla}) \chi = \dot{y}^\nu (\partial_\nu + i g A_\nu + \omega'_{\nu bc} \sigma^{bc}) \chi \quad (4.52)$$

$$\text{with } \omega'_{\nu bc} = +\frac{1}{2} e_b^\rho e_{c\rho;\nu} + \frac{1}{4} (e_{b\nu} e_c^\rho - e_{c\nu} e_b^\rho) \Lambda_{,\rho} \quad (4.53)$$

$$\text{and } e^{-\frac{\Lambda}{2}} \dot{\not{y}} = e^{-\frac{\Lambda}{2}} (\dot{y}^\nu e_{a\nu} \gamma^a) \quad (4.54)$$

Note that $e^\Lambda (\dot{y} \cdot \tilde{\nabla}) \chi$ and $e^{-\frac{\Lambda}{2}} \dot{\not{y}}$ are both Weyl invariants.

We now eliminate the auxiliary fields χ, Ψ from (4.51) and denote the resulting expression by X_{quark}^* . We have from the Euler-Lagrange equation for $\bar{\Psi}$

$$\frac{\delta X_{quark}}{\delta \bar{\Psi}} = 0 \quad \Rightarrow (\dot{y} \cdot \tilde{\nabla}) \chi = i [e^{\frac{\Lambda}{4}} (e^{-\frac{\Lambda}{2}} \dot{\not{y}}) \psi] \quad (4.55)$$

Again, we restrict the fields so that we have equality between the retarded and advanced solutions of (4.55). Thus,

$$((\dot{y} \cdot \tilde{\nabla})^{-1} \chi)(x) = \int_{-\infty}^0 V(x, x + Q\theta) \chi(x + Q\theta) = \int_0^\infty V(x, x + Q\theta) \chi(x + Q\theta)$$

$$\text{where } V(x, x + Q\theta) = \left(\text{P exp} - \int_{-\theta}^0 d\theta' [\dot{y}^\nu (i g A_\nu(y(x, \theta')) + \omega'_{\nu bc}(y(x, \theta')) \sigma^{bc})] \right) \quad (4.56)$$

$$= \left(\text{P exp} - \int_{-\theta'}^0 [i g \dot{y}^\nu A_\nu(y(x, \theta'))] \right) \cdot$$

$$\left(\text{P exp} - \int_{\theta'}^0 [\dot{y}^\nu \omega'_{\nu bc}(y(x, \theta')) \sigma^{bc}] \right)$$

where $y(x, \theta')$ is given by (2.17) subject to (2.15).

Noting that Ψ and $\bar{\Psi}$ are Lagrange multipliers, we can now eliminate the auxiliary fields. Using $[e^{-\frac{\Lambda}{2}} \dot{\not{y}}, (\dot{y} \cdot \tilde{\nabla})^{-1} e^\Lambda] \chi = 0$ for any χ (See Appendix D), we have the following expression for X_{quark}^*

$$X_{quark}^* = i a \int d^4x \sqrt{-g} (\bar{\psi} e^{\frac{3\Lambda}{4}} \dot{\not{y}} (\dot{y} \cdot \tilde{\nabla})^{-1} e^{\frac{\Lambda}{4}} \psi + \text{h. conj.}) \quad (4.57)$$

It is straightforward to verify that X_{quark}^* satisfies conditions (i)' and (ii)' of section (A.3) of Appendix A, where it is shown that this implies that $X_{quark}^* = X^{HTL(quark)}$ to $O(\kappa)$.

Therefore, we can use the same methods as before (section (2.3)) to calculate the quark energy momentum tensor :

$$\begin{aligned}
T_{\alpha\beta}^{HTL(quark)} &= \int \frac{d\Omega}{4\pi} S_{\alpha\beta}^{quark} \\
\text{with } S_{\alpha\beta}^{quark} &\equiv g^2 T^2 \lim_{e \rightarrow \eta} \left(e_\alpha^a \frac{\delta X_{quark}^*}{\delta e^{a\beta}} \right) = g^2 T^2 \lim_{e \rightarrow \eta} \left(e_\alpha^a \frac{\delta X^{HTL(quark)}}{\delta e^{a\beta}} \right) \\
S_{\alpha\beta}^{quark} &= a [Q_\alpha Q_\beta \square (Q \cdot \partial)^{-2} - Q_{\{\alpha} \partial_{\beta\}} (Q \cdot \partial)^{-1}] [\bar{\Psi}_0 \not{Q} \psi] \\
&+ a [Q_{\{\alpha} \eta_{\beta\}}^\lambda - Q_\alpha Q_\beta (Q \cdot \partial)^{-1} \partial^\lambda] [\bar{\Psi}_0 \gamma_\lambda \psi + i \bar{\Psi}_0 D_\lambda \not{Q} \Psi_0] \\
&+ \frac{i}{2} a [\partial_\mu (\bar{\Psi} Q_\alpha \sigma_\beta{}^\mu \chi + \bar{\Psi} Q_\beta \sigma_\alpha{}^\mu \chi + \bar{\Psi} Q^\mu \sigma_{\alpha\beta} \chi)] + h.conj.
\end{aligned} \tag{4.58}$$

It is straightforward to verify that, up to $O(\kappa)$, X_{quark}^* agrees with the quark gluon graviton effective action given by equation (34) of [6]. They do not agree at $O(\kappa^2)$ despite the fact that both satisfy conditions (i)' and (ii)' due to the fact that our definitions of Λ are different at $O(\kappa^2)$ (as discussed further in Appendix C).

Again, these calculations are valid in \mathcal{R} and may be analytically continued to fields off \mathcal{R} .

5 Conserved Currents

First of all we note that for any G_λ

$$\int d^3x G_0 = \int d^3x (Q \cdot \partial)^{-1} \partial^\lambda G_\lambda \tag{5.59}$$

Using (5.59) it is straightforward to calculate the integrated energy and momentum densities. Consider the gluon sector. We have from (2.32) (with $E = 0$)

$$\begin{aligned}
P_\alpha \equiv \int d^3x T_{0\alpha}^{HTL} &= \int \frac{d\Omega}{4\pi} d^3x 4 \text{tr} \left[\left(\frac{1}{2} \eta_{\alpha 0} \eta_{\mu\nu} - \eta_{\mu\alpha} \eta_{0\nu} \right) \left(\frac{3}{4} m_g^2 W_0^\mu W_0^\nu \right) \right. \\
&\quad \left. + \frac{1}{2} Q_\alpha \eta_0^\lambda (V_{0\mu} F^\mu{}_\lambda - V_{0\mu} D_\lambda W^\mu + \partial_\mu (V_0^\mu W_{0\lambda})) \right]
\end{aligned} \tag{5.60}$$

where W_0^μ and V_0^μ are given by (2.4). Notice that P_α is local before the elimination of the auxiliary fields. It is straightforward to verify that this expression agrees with P_α as would be calculated in other papers (compare eg. (3.13) of [2]) which Blaizot and Iancu [3], using properties of the angular integral $\int d\Omega (4\pi)^{-1}$, have shown to be equal to

$$P_0 = \int \frac{d\Omega}{4\pi} d^3x \frac{3}{4} m_g^2 [(Q \cdot D)^{-1} Q^\mu F_{0\mu}] [(Q \cdot D)^{-1} Q^\mu F_{0\mu}] \tag{5.61}$$

Thus P_0 is in fact a positive definite functional of the gluon fields even though this positivity is not manifest.

Similarly, we can write down P_α^{quark}

$$\begin{aligned}
P_\alpha^{quark} &\equiv \int d^3x T_{0\alpha}^{HTL(quark)} \\
&= \int \frac{d\Omega}{4\pi} \int d^3x a \left[-\eta_{\alpha 0} (\bar{\Psi}_0 \not{Q} \psi) + \bar{\Psi}_0 \gamma_\alpha \psi + i \bar{\Psi}_0 D_\alpha \not{Q} \Psi_0 \right] + h.conj.
\end{aligned} \tag{5.62}$$

where Ψ_0 is given by (4.45). We note that this expression is a local function of the auxiliary fields and does not appear to be positive definite which perhaps is to be expected eg. by comparison with the zero temperature quark contribution to energy momentum tensor. $T_{\alpha\beta}^{HTL}$ and $T_{\alpha\beta}^{HTL(quark)}$ are symmetric and traceless (on use of the equations of motion for $T^{HTL(quark)}$) (see [10]). Thus, the construction of angular momentum and scaling currents/charges is standard. They are defined by (for the gluon sector)

$$\begin{aligned} M_{\mu\lambda\nu} &= x_\mu(T_{\lambda\nu}^{HTL} - x_\nu T_{\lambda\mu}^{HTL}) & C_\mu &= x^\nu T_{\mu\nu}^{HTL} \\ J_{\mu\nu} &= \int d^3x M_{\mu 0\nu} & C &= \int d^3x C_0 \end{aligned} \quad (5.63)$$

with similar definitions for the quark sector expressions. It should be noted that these expressions unlike the P_μ are non local even before the elimination of the auxiliary fields.

As a final exercise, we compare $P_0 = \int d^3x T_{00}^{HTL}$ with the the Hamiltonian derived from the local auxiliary field lagrangian density \mathcal{L} where $g^2 T^2 X_0(Q; m_0, A, \eta) = \int d^4x \mathcal{L}$. We use the m_0^μ field model for simplicity as it has less constraints then the V_0^μ, W_0^μ model where V_0^μ acts as a lagrange multiplier.

In the following we write we write out colour indices (A, B, \dots) instead of using a matrix notation. For clarity, we also drop the flat space label $_0$ although we are working with flat space fields in this analysis. Thus all indices in the remainder of this section are either colour or coordinate ones.

$$\mathcal{L} = (2b)[m^{\lambda A} Q^\nu F_{\lambda\nu}^A + \frac{1}{2}(Q \cdot Dm_\nu^A)(Q \cdot Dm^{\nu A})] - \frac{1}{4}F_{\mu\nu}^A F^{\mu\nu A} \quad (5.64)$$

The conjugate momemnta for m and A are given respectively by

$$\Pi^{\mu A} = 2b(Q \cdot D)m^{\mu A} \quad \Pi_{gluon}^{\mu A} = E^{\mu A} + 2b(Q^\mu m^{0A} - m^{\mu A}) \quad (5.65)$$

where $E^{\mu A} = F^{\mu A}$. Hence

$$\partial_0 m^{\mu A} = \frac{1}{2b}\Pi^{\mu A} + \mathbf{Q} \cdot \mathbf{D}m^{\mu A} + if^{ABC}A_0^B m^{\mu C} \quad (5.66)$$

$$\begin{aligned} \partial_0 A_j^A &= D_j A_0^A - E_j^A & \Pi_{gluon}^{0A} &\equiv 0 \\ \mathcal{H} &\equiv \Pi_\mu^A \partial_0 m^{\mu A} + \Pi_{gluon}^{\mu A} \partial_0 A_\mu^A - \mathcal{L} \\ &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \Pi_\mu^A (\Pi^{\mu A} + \mathbf{Q} \cdot \mathbf{D}m^{\mu A} + if^{ABC}A_0^B m^{\mu C}) \\ &\quad - 2b m^{jA} Q^k F_{jk}^A + A_0^A (D_j \Pi_j^A) \end{aligned} \quad (5.67)$$

We want to compare $\int \frac{d\Omega}{4\pi} \int d^3x \mathcal{H}$ with $\int d^3x {}^m S_{00}^{E'=0}$.

[Note that in deriving ${}^m S_{\alpha\beta}^{E'=0}$ in (3.43) we have used the equations of motion for m to drop a term with a $(Q^\mu m_\mu)$ dependence, which we denote by $\delta({}^m S_{\alpha\beta}^{E'=0})$

$$\begin{aligned} \delta({}^m S_{\alpha\beta}^{E'=0}) &= -\frac{1}{2}[Q_\alpha Q_\beta \square (Q \cdot \partial)^{-2} + \eta_{\alpha\beta} - Q_{\{\alpha} \partial_{\beta\}} (Q \cdot \partial)^{-1}] \\ &\quad [\partial_\mu (m^\mu Q^\rho (Q \cdot Dm_\rho))] \end{aligned}$$

However it is straightforward to check that $\int d^3x \delta({}^m S_{00}^{E'=0}) = 0$ and hence $\int d^3x {}^m S_{00}^{E'=0}$ is given by integrating (3.43) with $E' = 0$].

Subtracting $\int \frac{d\Omega}{4\pi} \int d^3x \mathcal{H}$ from $\int d^3x {}^m S_{00}^{E'=0}$ we obtain not zero but

$$\int \frac{d\Omega}{4\pi} \int d^3x 2b {}^m 0^A [(Q \cdot D)^2 m^{0A} - Q^\mu F_{\mu}^{0A}] - A_0^A (D_j \Pi_j^A + i f^{BAC} \Pi_\mu^B m^{\mu C}) \quad (5.68)$$

Thus the m field theory produces a non standard result due to the unusual form of its ‘kinetic energy’ term $\frac{1}{2}(Q \cdot D m_\nu^A)(Q \cdot D m^{\nu A})$ which has cross terms of time and spatial derivatives. However, the physical theory is obtained by elimination of the auxiliary fields in which case the term proportional to m^0 drops out (see 3.35) leaving only the Gauss constraint term, ie. a gauge transformation generator which is constrained to zero when considering the physical, gauge invariant, theory. Hence, we see that for the physical theory $\int \frac{d\Omega}{4\pi} \int d^3x \mathcal{H} = \int d^3x T_{00}^{HTL}$.

6 Conclusions

In the above we have shown how to derive the gluon and quark contributions, $T_{\alpha\beta}^{HTL}$ and $T_{\alpha\beta}^{HTL(quark)}$, to the retarded QGP energy momentum tensor using an auxiliary field method. These tensors have some desirable properties :

- They are automatically gauge invariant, without having to add divergences ‘by hand’.
- They are symmetric. $T_{\alpha\beta}^{HTL} = T_{\beta\alpha}^{HTL}$ from its definition and $T_{\alpha\beta}^{HTL(quark)} = T_{\beta\alpha}^{HTL(quark)}$ as a consequence of the local lorentz symmetry and the equations of motion (See eg. [10]).
- $T_{\alpha}^{HTL}{}^{\alpha} = T_{\alpha}^{HTL(quark)}{}^{\alpha} = 0$ using Weyl scaling symmetry together with the equations of motion.
- $\partial^\alpha (T_{\alpha\beta}^{HTL} + T_{\alpha\beta}^{HTL(quark)}) = 0$, as required for a energy momentum tensor of a closed system. This is a consequence of general coordinate symmetry and the equations of motion. (Note that we have not introduced an external source in this paper and hence do not a external source current on the right hand side of the above, unlike Blaizot and Iancu).
- Despite appearances, $T_{\alpha\beta}^{HTL}$ is a positive definite functional of the gluon fields, although to show this requires properties of the angular integral ie. $\int d\Omega (4\pi)^{-1}$.

Properties (a) \rightarrow (c) allow straightforward construction of the conserved quantities of the QGP ie. momentum, angular momentum and the scaling current as given above.

However, the expressions for $T_{\alpha\beta}^{HTL}$ and $T_{\alpha\beta}^{HTL(quark)}$ are cumbersome, being non-local functionals even before elimination of the auxiliary fields. The angular momenta and scaling currents/charges are non local even before elimination of the auxiliary fields making their manipulation cumbersome too. Finally we conclude by noting that the insights gained using the above method have shown that $\Gamma^{HTL}[A, g]$ and $\Gamma^{HTL(quark)}[\bar{\psi}, \psi, A, g]$ are more complicated than previously thought and are certainly not known at present to $O(\kappa^2)$.

A Appendix A

In this Appendix we discuss the result that any expression, denoted $X^{guess}(Q; A, g]$ which

- i. Satisfies conditions (A) \rightarrow (D)
- ii. Reduces to $X_0^{HTL}(Q; A, \eta]$ when $g \equiv \eta$
- iii. Agrees with $X^{HTL}(Q; A, g]$ at $O(g^2\kappa)$

must in fact equal to $X^{HTL}(Q; A, g]$ at $O(\kappa)$ for all orders in g . In appendix C we briefly discuss the some of the further complications that arise at $O(\kappa^2)$.

Firstly, we note that this result is much weaker than previous speculation that (i) & (ii) would be sufficient to imply equality between $X^{guess}(Q; A, g]$ and $X^{HTL}(Q; A, g]$ [6, 7]. As seen at the end of section (2.2.3) X_E is a clear counterexample.

As condition (ii) gives equality at $O(\kappa^0)$ we are primarily interested in the theory at $O(\kappa)$. It is sufficient to consider

$$S_{\alpha\beta}^{guess} \equiv 2g^2 T^2 \lim_{g \rightarrow \eta} \left(\frac{\delta X^{guess}(Q; A, g]}{\delta g^{\alpha\beta}} \right) \quad (\text{A.69})$$

Suppose two different $X^{guess}(Q; A, g]$ exist satisfying (i) \rightarrow (iii) above, leading to two different $S_{\alpha\beta}^{guess}$ whose difference we denote by $\Delta S_{\alpha\beta}$.

We show below that (i) \rightarrow (iii) force $\Delta S_{\alpha\beta}$ to zero at all orders in g which proves the result stated at the start of this appendix.

A.1 Conditions (i) \rightarrow (iii) $\Rightarrow \Delta S_{\alpha\beta}=0$

Firstly note that (iii) implies that $\Delta S_{\alpha\beta}$ is zero at $O(g^2)$. The idea of the proof at higher orders in g is simple and in the same spirit as [7]. Write down the most general $O(g^s)$ [here $s \geq 3$] contribution to $\Delta S_{\alpha\beta}$ consistent with conditions (A) \rightarrow (C) of (i) above. Next, in a step that requires a lot of tedious but not difficult algebra, show that the homogeneous Ward identities that correspond to condition (D) of (i) above can only be satisfied if $\Delta S_{\alpha\beta}$ at this order is identically zero.

The most general $O(g^s)$ contribution to $\Delta S_{\alpha\beta}$, denoted $\Delta S_{\mu_{s+1}\mu_{s+2}}^s$, [where $\alpha \rightarrow \mu_{s+1}$ and $\beta \rightarrow \mu_{s+2}$] can be written as the following integral

$$\Delta S_{\mu_{s+1}\mu_{s+2}}^s = \int d^4 y_1 \dots d^4 y_s d^4 p_1 \dots d^4 p_s e^{-\sum_{i=1}^s p_i \cdot (y_i - x)} W_{\mu_1 \dots \mu_s \mu_{s+1} \mu_{s+2}}^s A^{\mu_1}(y_1) \dots A^{\mu_s}(y_s) \quad (\text{A.70})$$

where p_i ($i = 1 \dots s$) denote the i^{th} external gluon momentum. W^s is a function of $p_1 \dots p_s$ given by the expression below. First, some definitions (*used only in this appendix*).

$$\begin{aligned} Q_{\mu_i \mu_j \dots} &\equiv Q_{\mu_1} \dots Q_{\mu_{i-1}} Q_{\mu_{i+1}} \dots Q_{\mu_{j-1}} Q_{\mu_{j+1}} \dots Q_{\mu_{s+2}} \\ A_{ijkl}^* &\equiv A_{ijkl} \eta_{\mu_i \mu_j \mu_k \mu_l} Q_{\mu_i \mu_j \mu_k \mu_l} \quad B_{ij}^* \equiv B_{ij} \eta_{\mu_i \mu_j} Q_{\mu_i \mu_j} \\ C_{ijk}^* &\equiv (C_{ij})_{\mu_k} \eta_{\mu_i \mu_j} Q_{\mu_i \mu_j} \end{aligned}$$

where $A_{ijkl}, B_{ij}, (C_{ij})_{\mu_k}$ are (thusfar arbitrary) coefficients constructed from the gluon momenta and their contractions with Q_μ and each other. Note that the μ_k index of $(C_{ij})_{\mu_k}$ corresponds to an index on some combination of the external momenta (and thus it does not belong to a Q_μ or η). $D_{\mu_i\mu_j}$ and E_{μ_i} below are defined in the same way. Thus W^s can be written in form compatible with (A) \rightarrow (C) as

$$\begin{aligned}
W_{\mu_1 \dots \mu_{s+2}}^s &= \left\{ (A_{1234}^* + A_{1324}^* + A_{1423}^*) + (A_{1235}^* + A_{1325}^* + A_{1523}^*) + \dots \right. \\
&+ (A_{s-1,s,s+1,s+2}^* + A_{s-1,s+1,s,s+2}^* + A_{s,s+2,s-1,s+13}^*) \\
&+ B_{12}^* + \dots + B_{s+1,s+2}^* + (C_{123}^* + C_{231}^* + C_{321}^*) \\
&+ \dots + (C_{s-1,s,s+1}^* + C_{s,s+1,s-1}^* + C_{s+1,s,s-1}^*) + D_{\mu_1\mu_2} Q_{\overbrace{\mu_1\mu_2}} \\
&\left. + \dots + D_{\mu_{s+1}\mu_{s+2}} Q_{\overbrace{\mu_{s+1}\mu_{s+2}}} + E_{\mu_1} Q_{\overbrace{\mu_1}} + \dots + E_{\mu_{s+2}} Q_{\overbrace{\mu_{s+2}}} \right\}
\end{aligned} \tag{A.71}$$

The above expression is over-general in that it does not take into account all the symmetries of W^s , such as symmetry on interchange of graviton indices ($\mu_{s+1} \leftrightarrow \mu_{s+2}$) and, from the definition of W , symmetry under the interchange $(p_{\mu_i}, \mu_i) \leftrightarrow (p_{\mu_j}, \mu_j)$. These represent further constraints on A, \dots, E (eg. graviton indice symmetry requires $(A_{ijk,s+1} = A_{ijk,s+2} \ \& \ A_{i,s+1,j,s+2} = A_{i,s+2,j,s+1})$). However the Ward identities are sufficient to force A, \dots, E to zero even without such constraints and so these constraints are not explicitly incorporated into (A.71) for simplicity. We define

$$p_{s+1}^\mu = p_{s+2}^\mu = - \sum_{i=1}^s p_i^\mu \tag{A.72}$$

and thus $p_{s+1}^\mu = p_{s+2}^\mu$ give the momentum of the external graviton field.

We also define $P_i = p_i / (Q_\mu p_i^\mu)$ for $i = 1 \dots s+2$ and thus $Q_\mu P_i^\mu = 1$ for all i . Then the homogeneous Ward identities corresponding to gauge and general coordinate invariance of condition (D) are given by

$$P_i^{\mu_i} W_{\mu_1 \dots \mu_i \dots \mu_{s+2}} = 0 \quad i = 1 \dots s+2 \tag{A.73}$$

[Note that Weyl invariance implies a further homogeneous Ward identity that must be satisfied (which is not required in this section but will be used in the next subsection)]

$$\eta^{\mu_{s+1}\mu_{s+2}} W_{\mu_1 \dots \mu_{s+1}\mu_{s+2}} = 0 \tag{A.74}$$

Now apply the Ward identities given by (A.73) to (A.71). First examine the contribution which has (s-3) factors of Q_μ in the contraction of P_1 with W

$$\begin{aligned}
&[A_{2345}\eta_{\mu_2\mu_3}\eta_{\mu_4\mu_5} + A_{2435}\eta_{\mu_2\mu_4}\eta_{\mu_3\mu_5} + A_{2534}\eta_{\mu_2\mu_5}\eta_{\mu_3\mu_4}] Q_{\overbrace{\mu_1\mu_2\mu_3\mu_4\mu_5}} + \\
&[A_{2346}\eta_{\mu_2\mu_3}\eta_{\mu_4\mu_6} + A_{2436}\eta_{\mu_2\mu_4}\eta_{\mu_3\mu_6} + A_{2634}\eta_{\mu_2\mu_6}\eta_{\mu_3\mu_4}] Q_{\overbrace{\mu_1\mu_2\mu_3\mu_4\mu_6}} + \\
&\dots + \left\{ [A_{s-1,s,s+1,s+2}\eta_{\mu_{s-1}\mu_s}\eta_{\mu_{s+1}\mu_{s+2}} + A_{s-1,s+1,s,s+2}\eta_{\mu_{s-1}\mu_{s+1}}\eta_{\mu_s\mu_{s+2}} + \right. \\
&\quad \left. A_{s-1,s+2,s,s+1}\eta_{\mu_{s-1}\mu_{s+2}}\eta_{\mu_s\mu_{s+1}}] Q_{\overbrace{\mu_1\mu_{s-1}\mu_s\mu_{s+1}\mu_{s+2}}} \right\} = 0
\end{aligned}$$

However all terms above correspond to independent tensors and hence all the A_{ijkl} in the above expression must be zero. Hence all A_{ijkl} with none of $\{ijkl\}$ equal to one must be zero. Contracting with the other P_i shows that all A_{ijkl} must be zero. Now consider the contribution with (s-2) factors of Q_μ in the contraction of P_1 with W . This gives the B_{ij} (with i, j not equal to one) in terms of linear combinations of the A_{ijkl} which from above are zero and hence such B_{ij} must be zero too. Explicitly the P_1 contraction gives

$$B_{23}\eta_{\mu_2\mu_3}Q_{\overbrace{\mu_1\mu_2\mu_3}} + B_{24}\eta_{\mu_2\mu_4}Q_{\overbrace{\mu_1\mu_2\mu_4}} + \dots + B_{s+1,s+2}\eta_{\mu_s+1\mu_s+2}Q_{\overbrace{\mu_{s+1}\mu_{s+2}\mu_3}} \\ + \left\{ \text{Terms involving } A_{ijkl} = 0 \text{ from above} \right\} = 0$$

Considering all P_i contractions shows that all B_{ij} are zero. By examining contributions with increasing numbers of Q_μ it is straightforward to show that all $(C_{ij})_{\mu_k}$, $D_{\mu_i\mu_j}$ and E_{μ_i} are zero as well, which proves the desired result. Note that at $O(g^2)$, ie. the s=2 term, the above method is invalidated because in order to show the A_{ijkl} are zero it is required to consider terms with (s-3) factors of Q_μ .

We have now shown that the momentum space integrands of X^{HTL} and X^{guess} are the same at $O(\kappa)$. However, performing the momentum space integrals required to calculate X^{HTL} and X^{guess} is only well defined in Euclidean space for arbitrary fields. In Minkowski space the integrals are only defined for arbitrary fields via analytic continuation. However, for the fields belonging to \mathcal{R} , the support of the fields' Fourier transforms is sufficiently restricted for the Minkowski momentum space integrals to be well defined even without analytic continuation. Hence for fields in \mathcal{R} agreement of momentum space integrands gives equality of X^{HTL} and X^{guess} . [For fields not in \mathcal{R} , the same analytic continuation would have to be used for both X^{HTL} and X^{guess} to ensure equality].

A.2 Relaxing condition (iii) above

We also briefly remark that if condition (iii) above is relaxed then $\Delta S_{\alpha\beta}$ is not forced to zero by conditions (i) and (ii). The most general form of $W_{\mu_1\mu_2\mu_3\mu_4}^{s=2}$ taking into account the further constraints mentioned above is written down as $W_{\alpha\beta}^{\mu\nu}(P, R, K)$ as given by equation (B1) of [7] noting the following notational changes

$$([P_1, \mu_1] ; [P_2, \mu_2] ; [P_3, \mu_3] ; [P_4, \mu_4]) \leftrightarrow ([R, \mu] ; [-P, \nu] ; [K, \alpha] ; [K, \beta])$$

Of course, this agrees with (A.71) for s=2 on imposing the additional constraints discussed above. Following the analysis of [7] we find that equation (B2) of [7] holds but equation (B3) of [7] is erroneous. Writing all the equations required to explicitly show this is tedious and not very informative. However the end result, contrary to the erroneous conclusion of [7], is that the most general form of the $O(g^2)$ contribution to $\Delta S(C)_{\alpha\beta}$ consistent with the Ward identities is given by

$$\Delta S(C)_{\alpha\beta} = (gT)^2 \int d^4y d^4y' d^4p d^4r e^{-ip \cdot (y-x) + ir \cdot (y'-x)} A_\mu(y) [C W_{\alpha\beta}^{\mu\nu}(p, r)] A_\nu(y') \quad (\text{A.75})$$

with

$$\begin{aligned}
W_{\alpha\beta}^{\mu\nu} = & [(Q^\nu P_\alpha - \eta_\alpha^\nu)(Q^\mu R_\beta - \eta_\beta^\mu) + (Q^\mu P_\beta - \eta_\beta^\mu)(Q^\nu R_\alpha - \eta_\alpha^\nu)] \\
& - [Q_{\{\alpha}\eta_{\beta\}}^\lambda - Q_\alpha Q_\beta K^\lambda] K^\rho [(Q^\nu P_\rho - \eta_\rho^\nu)(Q^\mu R_\lambda - \eta_\lambda^\mu) + (P, \nu) \rightleftharpoons (R, \mu)] \\
& - [Q_\alpha Q_\beta K^2 + \eta_{\alpha\beta} - Q_{\{\alpha} K_{\beta\}}][(Q^\nu P_\rho - \eta_\rho^\nu)(Q^\mu R^\rho - \eta^{\mu\rho})]
\end{aligned} \tag{A.76}$$

and C is either a constant or possibly a scalar dimensionless and rational function of the $\{(Q \cdot p), (Q \cdot r), (Q \cdot k)\}$ as multiplication by any other function will contradict (A) \rightarrow (D) of condition (i) above.

Note that the coefficient of (2.32) can be written in the form of (A.75) for $C = (Q \cdot k)^2 / [(Q \cdot p)(Q \cdot r)]$. This shows that the auxiliary field construction of section 2, which for $E \neq 0$ satisfies (i) and (ii) but not (iii), is consistent with the above. The same applies for section 3.

A.3 The Quark Sector

In this subsection we outline a proof of the following: any expression, denoted $X^{guess(quark)}(Q; \bar{\psi}, \psi, A, e]$, which

- i. Satisfies conditions (A) \rightarrow (D)
- ii. Reduces to $X_0^{HTL}[A, \eta]$ when $e \equiv \eta$

must in fact equal to $X^{HTL(quark)}(Q; \bar{\psi}, \psi, A, e]$ at $O(\kappa)$ for all orders in g which is consistent with the assertions made in [6]. We proceed in the same way as above and first of all define $S_{\alpha\beta}^{guess(quark)}$ by

$$S_{\alpha\beta}^{guess(quark)} \equiv g^2 T^2 \lim_{e \rightarrow \eta} \left(e_\alpha^a \frac{\delta X^{guess(quark)}(Q; \bar{\psi}, \psi, A, e]}{\delta e^{a\beta}} \right) \tag{A.77}$$

Again, we suppose two different $X^{guess}(Q; A, e]$ exist satisfying (i) \rightarrow (iii) above, leading to two different $S_{\alpha\beta}^{guess}$ whose difference we denote by $\Delta S_{\alpha\beta}^{(quark)}$. The most general form of the $O(g^s)$ contribution to $\Delta S_{\alpha\beta}^{(quark)}$ can be written as follows (with $\alpha \rightarrow \mu_{s+1}$ and $\beta \rightarrow \mu_{s+2}$)

$$\begin{aligned}
\Delta S_{\mu_{s+1}\mu_{s+2}}^s &= \int d^4 z_1 d^4 z_2 d^4 y_1 \dots d^4 y_{s+2} d^4 r_1 d^4 r_2 d^4 p_1 \dots d^4 p_s \\
& e^{-\sum_{i=1}^s p_i \cdot (y_i - x) - \sum_{j=1}^2 r_j \cdot (z_j - x)} \bar{\psi}(z_1) W_{\mu_1 \dots \mu_s \mu_{s+1} \mu_{s+2}}^{s(quark)} \psi(z_2) A^{\mu_1}(y_1) \dots A^{\mu_s}(y_s)
\end{aligned} \tag{A.78}$$

where $W^{s(quark)}$ is a function of r_1, r_2, p_1, p_{s+2} which are the momenta of, respectively, the anti-quark, quark and the s gluon fields. μ_{s+1}, μ_{s+2} are the vierbien indices. The most general form of $W^{s=0(quark)}$ consistent with conditions (A) \rightarrow (C) is given by

$$\begin{aligned}
W_{\mu_1 \dots \mu_{s+2}}^{s(quark)} &= Q_{\mu_1} \dots Q_{\mu_{s+2}} [A' + A \not{Q}] + \sum_{j=1}^{s+2} Q_{\mu_j} \frown_{\mu_j} [B_j' \gamma_{\mu_j} + (B_{\mu_j}^j + C_j \gamma^{\mu_j}) \not{Q}] \\
&+ \sum_{s+2 \geq j > k \geq 1} Q_{\mu_j \mu_k} \frown_{\mu_j \mu_k} [(D_{jk} \gamma_{\mu_j} \gamma_{\mu_k} + E_{jk} \eta_{\mu_j \mu_k}) \not{Q}]
\end{aligned} \tag{A.79}$$

where $A', A, B'_j, C_j, D_{jk}, E_{jk}$ are coefficients formed from the various contractions of the external momenta with the following possibilities: the external momenta, the gamma matrices or the Q_μ . (Note that they contain no \mathcal{Q}). The $B_{\mu_j}^j$ are proportional to some linear combination of the external momenta. The exact form of the above coefficients is heavily restricted by conditions $(A) \rightarrow (C)$.

Note that due to the relation $\gamma_{\mu_2}\gamma_{\mu_1} = -\gamma_{\mu_1}\gamma_{\mu_2} + 2\eta_{\mu_1\mu_2}$ the most general of form of $W^{s=0(quark)}$ can be written without terms proportional to $\gamma_{\mu_j}\gamma_{\mu_k}$ with $j \leq k$ and also with all \mathcal{Q} dependence written with a single \mathcal{Q} exclusively on the right hand side of (A.79).

Now consider the homogeneous Ward identities. First we define

$$p_{s+1}^\mu = p_{s+2}^\mu = -\sum_{i=1}^s p_i^\mu - r_1^\mu - r_2^\mu \quad (\text{A.80})$$

and hence $p_{s+1}^\mu = p_{s+2}^\mu$ gives the momentum of the external vierbien field. Then with the definition $P_i^\mu = (Q \cdot p_i)^{-1} p_i^\mu$ for $i = 1 \dots s+2$ the homogeneous Ward identities are given by

$$P^{\mu_j} W_{\mu_1 \dots \mu_j \dots \mu_{s+2}}^{s=0(quark)} = 0 \quad \text{for } j = 1 \dots s+2 \quad (\text{A.81})$$

$$\eta^{\mu_{s+1}\mu_{s+2}} W_{\mu_1 \dots \mu_{s+1}\mu_{s+2}}^{s=0(quark)} = 0 \quad W_{\mu_1 \dots \mu_{s+1}\mu_{s+2}}^{s(quark)} = W_{\mu_1 \dots \mu_{s+2}\mu_{s+1}}^{s(quark)} \quad (\text{A.82})$$

For the $s = 0$ case, it is just an exercise in linear algebra to show that these identities force $W_{\mu_1\mu_2}^{s=0(quark)}$ to zero. Thus we have the equality of $X^{HTL(quark)}$ and $X^{guess(quark)}$ at $O(g^2\kappa)$ (or rather equality of their momentum space integrands).

The method of determining equality at higher orders in g is now identical to the method of the gluon sector i.e. consider the Ward identities given by (A.81) and the coefficients of the terms with increasing numbers of Q^μ factors. This is sufficient to force $W_{\mu_1 \dots \mu_{s+2}}^{s(quark)}$ to zero as in the gluon case.

Thus we have equality between the angular integrands of $X^{HTL(quark)}$ and $X^{guess(quark)}$ at $O(\kappa)$ for all orders in g and thus $X^{HTL(quark)} = X^{guess(quark)}$ at $O(\kappa)$. [See remarks at the end of section (A.1)].

B Appendix B

The generalisation of the effective action to include the interaction of weakly coupling gravitons is independent (at $O(\kappa)$) of the Weyl scaling properties of the auxiliary fields. Suppose, as in section 2, we have the auxiliary fields $V(x, Q)$ and $W(x, Q)$ transforming on Weyl scaling as $W_\mu \rightarrow e^{2u\sigma} W_\mu$, and $V \rightarrow e^{2v\sigma} V_\mu$ with

Then, the Weyl invariant extension becomes

$$\begin{aligned} g^2 T^2 X_{E,u,v} &\equiv \int d^4x (\sqrt{-g} g^{\mu\nu} e^\Lambda) 2\text{tr} \left[-e^{2u\Lambda} \frac{3}{4} m_g^2 W_\mu W_\nu \right. \\ &\quad + e^{-v\Lambda} V_\mu \dot{y}^\lambda F_{\nu\lambda} - V_\nu [\dot{y}^\alpha \nabla_\alpha (e^{-u\Lambda} W_\mu + e^{-u\Lambda} (W_\nu \nabla_\mu \dot{y}^\nu - \dot{y}^\alpha W_\alpha \Lambda_{,\mu}))] \\ &\quad \left. + E \left\{ e^{-(u+v)\Lambda} V_\nu [W_\lambda \nabla_\mu \dot{y}^\lambda - \dot{y} \cdot W \partial \Lambda_\mu + g_{\mu\lambda} W \cdot \nabla \dot{y}^\lambda + W_\mu \dot{y} \cdot \partial \Lambda - \dot{y}_\mu W_\alpha \Lambda^{,\alpha}] \right\} \right] \\ \Gamma_E &\equiv g^2 T^2 \int \frac{d\Omega}{4\pi} X_E(Q; V, W, A, g) \end{aligned} \quad (\text{B.83})$$

Notice that at $O(\kappa)$ the coefficient of E does not change as $\Lambda \sim \nabla \dot{y} \sim O(\kappa)$.

Thus, we have

$$\begin{aligned} X_{E,u,v}(Q; V, W, A, g) &= X_{E,u=0,v=0}(Q; e^{-v\Lambda}V, e^{-u\Lambda}W, A, g) \\ &= X_{E,0,0}(Q; V, W, A, g) - \kappa \int dx \Lambda (uV \cdot \frac{\delta X}{\delta V} + vW \cdot \frac{\delta X}{\delta W}) + O(\kappa^2) \end{aligned}$$

Note that the $O(\kappa)$ term on the right of the above vanishes at solutions of the auxiliary field Euler-Lagrange equations. Hence on eliminating the auxiliary fields by use of the Euler-Lagrange equations this term automatically drops out. Thus we have equality at ($O(\kappa)$) between the two angular integrals on elimination of the auxiliary fields. This idea is easily applied to the other angular integrands of sections 3 and 4 as well.

C Appendix C

In this appendix we discuss further ambiguities that occur in attempting to write down $\Gamma^{HTL}[A, g]$ and $\Gamma^{HTL(quark)}[A, e]$ to $O(\kappa^2)$ using their known flatspace contributions and their symmetry properties given by (A) \rightarrow (D) above. Firstly consider Λ , defined to be a scalar which vanishes when $g = \eta$ with the Weyl scaling property that $\Lambda \rightarrow \Lambda + 2\sigma$.

This does not fix Λ uniquely at $O(\kappa^2)$ and there are at least two degrees of freedom in Λ at this order. Consider $\Lambda(a_1, a_2, x)$ defined by

$$\begin{aligned} \Lambda(a_1, a_2, x) &= - \int_0^\infty d\theta e^{\Lambda(a_1, a_2, y(x, \theta))} \int_0^\infty d\theta' e^{\Lambda(a_1, a_2, y(x, \theta'))} \left\{ e^{-2\Lambda(a_1, a_2, y(x, \theta'))} \right. \\ &\quad \left. \left[R_{\mu\nu}(y(x, \theta')) \dot{y}^\mu(x, \theta') \dot{y}^\nu(x, \theta') + \frac{1}{2} \frac{d\Lambda(a_1, a_2, y(x, \theta'))^2}{d\theta'} + a_1 J_1 + a_2 J_2 \right] \right\} \\ \text{where } J_1 &= \frac{1}{2} \nabla_\alpha \dot{y}^\alpha(x, \theta') \nabla_\beta \dot{y}^\beta(x, \theta') - \nabla_\alpha \dot{y}^\beta(x, \theta') \nabla_\beta \dot{y}^\alpha(x, \theta') \\ J_2 &= \nabla_\beta \dot{y}^\alpha(x, \theta') \nabla^\beta \dot{y}_\alpha(x, \theta') - \nabla_\beta \dot{y}^\alpha(x, \theta') \nabla_\alpha \dot{y}^\beta(x, \theta') \end{aligned}$$

It is easily confirmed that that on Weyl scaling J_1 and J_2 both merely scale by $e^{4\sigma}$ and that $\int d\theta e^{\Lambda(a_1, a_2, y(x, \theta))}$ is a Weyl invariant if $\Lambda \rightarrow \Lambda + 2\sigma$ on Weyl scaling [as a null geodesic, although invariant, is reparameterised on Weyl scaling with an affine parameter, θ , transforming as $d\theta \rightarrow e^{-2\sigma(\theta)} d\theta$].

Thus, using the well known (see eg. [11]) Weyl scaling properties of $R_{\mu\nu}$, it can be shown that on Weyl scaling

$$\Lambda(a_1, a_2, x) \rightarrow \Lambda(a_1, a_2, x) + 2\sigma(x) \quad (\text{C.84})$$

for arbitrary a_1 and a_2 . Thus we see that there is at least a further 2-parameter ambiguity at $O(\kappa^2)$ in trying to determine $\Gamma^{HTL}[A, g]$ and $\Gamma^{HTL(quark)}[A, e]$.

Note that $\Lambda(0, 0, x)$ is the same as the expression used for Λ in [6] [as given by equation(21) on using equation (55)] whereas $\Lambda(1, 0, x)$ reduces to $\Lambda = + \int_0^\infty d\theta' \nabla_\nu \dot{y}^\nu|_{y(x, \theta')}$ ie. the Λ used throughout this paper.

We also note that the local auxiliary field Lagrangians might well be dependent at $O(\kappa^2)$ on the Weyl scaling properties of the auxiliary fields suggesting the presence of further ambiguities at $O(\kappa^2)$.

D Appendix D

Here we show that for any χ

$$[e^{-\frac{\Lambda}{2}}\dot{\psi}, (\dot{y} \cdot \tilde{\nabla})^{-1}e^\Lambda]\chi = 0 \quad (\text{D.85})$$

where we recall that

$$(\dot{y} \cdot \tilde{\nabla})\chi = \dot{y}^\nu(\partial_\nu + igA_\nu + \omega'_{\nu bc}\sigma^{bc}) \quad (\text{D.86})$$

$$\text{with } \omega'_{\nu bc} = +\frac{1}{2}e_b^\rho e_{c\rho;\nu} + \frac{1}{4}(e_{b\nu}e_c^\rho - e_{c\nu}e_b^\rho)\Lambda_{,\rho} \quad (\text{D.87})$$

It is sufficient to show that

$$[e^{-\frac{\Lambda}{2}}\dot{\psi}, (\dot{y} \cdot \tilde{\nabla})]\chi = 0 \quad (\text{D.88})$$

(as $(\dot{y} \cdot \tilde{\nabla})$ is invertible, we can write $\chi = (\dot{y} \cdot \tilde{\nabla})^{-1}\Psi$ and then operating on (D.88) by $(\dot{y} \cdot \tilde{\nabla})^{-1}e^\Lambda$ gives (D.85) for arbitrary Ψ .)

Note that the following will be useful in going from the first to the second line of the calculation below.

$$[\gamma^a, \omega'_{\nu bc}\sigma^{bc}] = +2\omega'_\nu{}^a{}_c\gamma^c \quad e_{a\alpha}\dot{y}^\mu\partial_\mu\dot{y}^\alpha = -\dot{y}^\mu\dot{y}^\lambda\Gamma_{\mu\lambda}^\alpha e_{a\alpha} \quad (\text{D.89})$$

where the right hand equation is due to the geodesic equation for \dot{y}^μ . Therefore

$$\begin{aligned} [e^{-\frac{\Lambda}{2}}\dot{\psi}, (\dot{y} \cdot \tilde{\nabla})]\chi &= [e^{-\frac{\Lambda}{2}}\dot{y}^\lambda e_{a\lambda}\gamma^a, \dot{y}^\mu(\partial_\mu + \omega'_{\nu bc}\sigma^{bc})]\chi \\ &= e^{-\frac{\Lambda}{2}}\dot{y}^\mu\dot{y}^\lambda\gamma^c[\Gamma_{\mu\lambda}^\alpha e_{c\lambda} + \frac{1}{2}\Lambda_{,\mu}e_{c\lambda} - e_{c\lambda,\mu} + 2e_{a\lambda}\omega'_\nu{}^a{}_c\gamma^c]\chi \\ &= e^{-\frac{\Lambda}{2}}\dot{y}^\mu\dot{y}^\lambda\gamma^c[-(e_{c\lambda;\mu} - 2e_{a\lambda}\{\frac{1}{2}e^{a\rho}e_{c\rho;\mu}\}) \\ &\quad + \frac{1}{2}(e_{c\lambda}\Lambda_{,\mu} - e_{c\mu}\Lambda_{,\lambda} + g_{\mu\lambda}e_c^\rho\Lambda_{,\rho})] \end{aligned}$$

Terms involving Λ drop out (note that $g_{\mu\lambda}\dot{y}^\mu\dot{y}^\lambda = 0$) and it is easy to check that the remaining term is identically zero.

Hence $[e^{-\frac{\Lambda}{2}}\dot{\psi}, (\dot{y} \cdot \tilde{\nabla})]\chi = 0$ as required.

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References

- [1] E. Brateen, R. Pisarski, Nucl. Phys. **B337** (1990) 569

- [2] H. A. Weldon, Canadian J. Phys. **71** (1993) 300.
- [3] J.P. Blaizot and E. Iancu, Saclay Preprint T94/03 (Jan 94)
- [4] J.P. Blaizot and E. Iancu, Nucl. Phys. **B**, (In press)
(Soft collective excitations in hot gauge theories)
- [5] F.T Brandt, J. Frenkel and J.C. Taylor, Nucl. Phys. **B374** (1992) 169
- [6] F.T Brandt, J. Frenkel and J.C. Taylor, Nucl. Phys. **B410** (1993) 3.
- [7] F.T Brandt, J. Frenkel and J.C. Taylor, Nucl. Phys. **B410** (1993) 3, Appendix B.
- [8] R. Efraty and V. P. Nair Phys. Rev. **D47** (5601) 1993
- [9] R. Jackiw and V.P Nair Phys. Rev. **D48** (1993) 4991
- [10] S. Weinberg, Gravitation And Cosmology (John Wiley & Sons, New York, 1972)
Chapter 12
- [11] S. Hawking and G.F.R. Ellis, Large Scale Structure Of Space-time (Cambridge University Press, 1973) p. 42.

Note that $e^{-\sigma}$ in my notation is given as Ω in the above authors' notation.